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LOCALIZED BMO AND BLO SPACES ON RD-SPACES AND APPLICATIONS TO SCHRÖDINGER OPERATORS

DACHUN YANG, DONGYONG YANG AND YUAN ZHOU

School of Mathematical Sciences, Beijing Normal University,

Laboratory of Mathematics and Complex systems, Ministry of Education,

Beijing 100875, People's Republic of China

ABSTRACT. An RD-space $\mathcal{X}$ is a space of homogeneous type in the sense of Coifman and Weiss with the additional property that a reverse doubling condition holds in $\mathcal{X}$. Let ρ be an admissible function on RD-space $\mathcal{X}$. The authors first introduce the localized spaces $\mathrm{BMO}_\rho(\mathcal{X})$ and $\mathrm{BLO}_\rho(\mathcal{X})$ and establish their basic properties, including the John-Nirenberg inequality for $\mathrm{BMO}_\rho(\mathcal{X})$, several equivalent characterizations for $\mathrm{BLO}_\rho(\mathcal{X})$, and some relations between these spaces. Then the authors obtain the boundedness on these localized spaces of several operators including the natural maximal operator, the Hardy-Littlewood maximal operator, the radial maximal functions and their localized versions associated to ρ, and the Littlewood-Paley g-function associated to ρ, where the Littlewood-Paley g-function and some of the radial maximal functions are defined via kernels which are modeled on the semigroup generated by the Schrödinger operator. These results apply in a wide range of settings, for instance, to the Schrödinger operator or the degenerate Schrödinger operator on $\mathbb{R}^d$, or the sub-Laplace Schrödinger operator on Heisenberg groups or connected and simply connected nilpotent Lie groups.

1. Introduction. Since the space, $\mathrm{BMO}(\mathbb{R}^d)$, of functions with bounded mean oscillation on $\mathbb{R}^d$ was introduced by John and Nirenberg [20], it then plays an important role in harmonic analysis and partial differential equations. For example, it is well known that $\mathrm{BMO}(\mathbb{R}^d)$ is the dual space of the Hardy space $H^1(\mathbb{R}^d)$ (see, for example, [27, 13]), and also a good substitute of $L^\infty(\mathbb{R}^d)$. Recall that the Riesz transforms $\nabla(-\Delta)^{-1/2}$ are bounded on $\mathrm{BMO}(\mathbb{R}^d)$ but not on $L^\infty(\mathbb{R}^d)$ (see again, for example, [27, 13]), where $\Delta \equiv \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$ is the Laplacian and ∇ is the gradient operator. However, the space $\mathrm{BMO}(\mathbb{R}^d)$ is essentially related to the Laplacian Δ.

Let $\mathcal{L} \equiv -\Delta + V$ be the Schrödinger operator on $\mathbb{R}^d$, where the potential V is a nonnegative locally integrable function. Recently, there is an increasing interest on the study of these operators. In particular, Fefferman [10], Shen [26] and Zhong [34] established some

basic results, including some estimates of the fundamental solutions and the boundedness on Lebesgue spaces of Riesz transforms, for $\mathcal{L}$ on $\mathbb{R}^d$ with $d \geq 3$ and the nonnegative potential V satisfying the reverse Hölder inequality. Especially, the works of Shen [26] lay the foundation for developing harmonic analysis related to $\mathcal{L}$ on $\mathbb{R}^d$. Li [21] extended part of these results in [26] to the sub-Laplace Schrödinger operator on connected and simply connected nilpotent Lie groups. On the other hand, denote by $\mathcal{B}_q(\mathbb{R}^d)$ the class of functions satisfying the reverse Hölder inequality of order q. For $V \in \mathcal{B}_{d/2}(\mathbb{R}^d)$ with $d \geq 3$, Dziubański et al [9] introduced the BMO-type space $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^d)$ associated to the auxiliary function ρ determined by the potential V (see, for example, (4) below) and established the duality between $H_{\mathcal{L}}^1(\mathbb{R}^d)$ and $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^d)$, as well as a characterization of $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^d)$ in terms of the Carleson measure and the $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^d)$ boundedness of the variants of some classical operators associated to $\mathcal{L}$ including semigroup maximal functions and the Hardy-Littlewood maximal function. These results were generalized to Heisenberg groups by Lin and Liu [22]. Also, it is now known that $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^d)$ in [9] is a special case of BMO-type spaces introduced by Duong and Yan [4, 5]; see, in particular, [5, Proposition 6.11] and also [33].

Recently, a theory of Hardy spaces and their dual spaces on so-called RD-spaces was established in [15, 16, 14]. A space of homogenous type $\mathcal{X}$ in the sense of Coifman and Weiss ([2, 3]) is called to be an RD-space if $\mathcal{X}$ has the additional property that a reverse doubling condition holds in $\mathcal{X}$ (see [16]). It is well known that a connected space of homogeneous type is an RD-space. Typical examples of RD-spaces include Euclidean spaces, Euclidean spaces with weighted measures satisfying the doubling property, Heisenberg groups, connected and simply connected nilpotent Lie groups ([29, 30]) and the boundary of an unbounded model polynomial domain in $\mathbb{C}^N$ ([24]), or more generally, Carnot-Carathéodory spaces with doubling measures ([25, 16]). In [31], modeled on the known auxiliary function determined by V, a notion of admissible functions ρ was introduced and a theory of the localized Hardy space $H_\rho^1(\mathcal{X})$ associated with a given admissible function ρ was developed. In particular, the space $H_\rho^1(\mathcal{X})$ was characterized via several maximal functions modeled on the semigroup maximal operators generated by Schrödinger operators, including the localized radial maximal function S_ρ^+.

One of the main purposes of this paper is to investigate behaviors of these maximal operators aforementioned on localized BMO spaces. Precisely, let ρ be an admissible function on RD-space $\mathcal{X}$. We first introduce the localized BMO space $\mathrm{BMO}_\rho(\mathcal{X})$ and localized BLO space $\mathrm{BLO}_\rho(\mathcal{X})$, and establish their basic properties, including the John-Nirenberg inequality for $\mathrm{BMO}_\rho(\mathcal{X})$, several equivalent characterizations for $\mathrm{BLO}_\rho(\mathcal{X})$, and some relations between these spaces. Then we obtain the boundedness on these localized spaces of several operators including the natural maximal operator, the Hardy-Littlewood maximal operator, the radial maximal functions and their localized versions associated to ρ, and the Littlewood-Paley g-function associated to ρ, where the Littlewood-Paley g-function and some of the radial maximal functions are defined via kernels which are modeled on the semigroup generated by the Schrödinger operator. These results apply in a wide range of settings. Moreover, even when these results are applied, respectively, to the Schrödinger operator or the degenerate Schrödinger operator on $\mathbb{R}^d$, or the sub-Laplace Schrödinger operator on Heisenberg groups or connected and simply connected nilpotent

Lie groups, we also obtain some new results.

To be precise, this paper is organized as follows.

In Section 2, we first recall some notation and notions from [16, 31], including the approximation of the identity, the admissible function ρ, the radial maximal function $S^+(f)$ and the localized radial maximal function $S_\rho^+(f)$, where $S^+(f)$ and $S_\rho^+(f)$ are defined via a given approximation of the identity.

In Section 3, letting ρ be an admissible function on $\mathcal{X}$, we first introduce the localized BMO space $\mathrm{BMO}_\rho(\mathcal{X})$ and localized BLO space $\mathrm{BLO}_\rho(\mathcal{X})$; see Definitions 3.1 and 3.2 below. We also recall the notions of their global versions in Definitions 3.1 and 3.2 below. Then we establish some useful properties concerning these spaces, including the John-Nirenberg inequality for $\mathrm{BMO}_\rho(\mathcal{X})$ (see Theorem 3.1 below), several characterizations and inclusion relations of these spaces (see Lemma 3.1, Remarks 3.1 and 3.2, and Corollary 3.1 below). Then we prove that the function in $\mathrm{BLO}_\rho(\mathcal{X})$ has lower bound in Theorem 3.2, and establish several equivalent characterizations of $\mathrm{BLO}_\rho(\mathcal{X})$ in Theorems 3.2 and 3.3, Remark 3.3, and Corollaries 3.2 and 3.3 below.

In Section 4, we establish the boundedness of the natural maximal function, the Hardy-Littlewood maximal function and their localized versions from $\mathrm{BMO}_\rho(\mathcal{X})$ to $\mathrm{BLO}_\rho(\mathcal{X})$, and as an application, we obtain several equivalent characterizations for $\mathrm{BLO}_\rho(\mathcal{X})$ via the localized natural maximal function; see Theorems 4.1 and 4.2, Lemma 4.1 and Corollary 4.1 below. We point out that Corollary 4.1 improves the results of [9] and [22] even for the Schrödinger operators on $\mathbb{R}^d$ or Heisenberg groups with the potentials satisfying certain reverse Hölder inequality; see Remark 4.1 below.

In Section 5, we establish the boundedness of some maximal operators from $\mathrm{BMO}_\rho(\mathcal{X})$ to $\mathrm{BLO}_\rho(\mathcal{X})$. To be precise, the boundedness of the radial maximal functions $S^+(f)$, $S_\rho^+(f)$ and certain maximal operator T^+ from $\mathrm{BMO}_\rho(\mathcal{X})$ to $\mathrm{BLO}_\rho(\mathcal{X})$ are presented in Section 5.1; see Theorem 5.1, Corollaries 5.1 and 5.2 below. These operators were used, respectively in [14] and [31], to characterize the corresponding Hardy spaces $H^1(\mathcal{X})$ and $H_\rho^1(\mathcal{X})$. Section 5.2 is devoted to the boundedness of P^+ from $\mathrm{BMO}_\rho(\mathcal{X})$ to $\mathrm{BLO}_\rho(\mathcal{X})$; see Theorem 5.2 below. Here, T^+ and P^+ are defined via kernels which are modeled on the semigroup generated by the Schrödinger operator, and were used in [31] to characterize the corresponding Hardy space $H_\rho^1(\mathcal{X})$.

In Section 6, we obtain the boundedness on $\mathrm{BMO}_\rho(\mathcal{X})$ of the Littlewood-Paley g-function which is also defined via kernels modeled on the semigroup generated by the Schrödinger operator. Assuming that g-function is bounded on $L^2(\mathcal{X})$, we prove that if $f \in \mathrm{BMO}_\rho(\mathcal{X})$, then $[g(f)]^2 \in \mathrm{BLO}_\rho(\mathcal{X})$ with norm no more than $C\|f\|_{\mathrm{BMO}_\rho(\mathcal{X})}^2$, where C is a positive constant independent of f; see Theorem 6.1 below. As a corollary, we obtain the boundedness of the Littlewood-Paley g-function from $\mathrm{BMO}_\rho(\mathcal{X})$ to $\mathrm{BLO}_\rho(\mathcal{X})$; see Corollary 6.1 below.

In Section 7, we apply results obtained in Sections 5 and 6, respectively, to the Schrödinger operator or the degenerate Schrödinger operator on $\mathbb{R}^d$, the sub-Laplace Schrödinger operator on Heisenberg groups or on connected and simply connected nilpotent Lie groups. The nonnegative potentials of these Schrödinger operators are assumed to satisfy the reverse Hölder inequality. See Propositions 7.2, 7.3, 7.4 and 7.5 below. Even for these special cases, our results further improve and generalize the corresponding results

in [9, 22].

We now make some conventions. Throughout this paper, we always use C or A to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as C_1 or A_1, do not change in different occurrences. If $f \leq Cg$, we then write $f \lesssim g$ or $g \gtrsim f$; and if $f \lesssim g \lesssim f$, we then write $f \sim g$. For any given "normed" spaces $\mathcal{A}$ and $\mathcal{B}$, the symbol $\mathcal{A} \subset \mathcal{B}$ means that for all $f \in \mathcal{A}$, then $f \in \mathcal{B}$ and $\|f\|_{\mathcal{B}} \lesssim \|f\|_{\mathcal{A}}$. We always use B to denote a ball of $\mathcal{X}$, and for any ball $B \subset \mathcal{X}$, we denote by x_B the center of B, r_B the radius of B, and $B^\complement \equiv \mathcal{X} \setminus B$. Moreover, for any ball $B \subset \mathcal{X}$ and $\lambda > 0$, we denote by λB the ball centered at x_B and having radius λr_B. Also, χ_E denotes the characteristic function of any set $E \subset \mathcal{X}$. For all $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ and balls B, we always set $f_B \equiv \frac{1}{\mu(B)} \int_B f(y) \, d\mu(y)$.

2. **Preliminaries**. We first recall the notions of spaces of homogeneous type in the sense of Coifman and Weiss [2, 3] and RD-spaces in [16].

Definition 2.1. Let $(\mathcal{X}, d)$ be a metric space with a regular Borel measure μ such that all balls defined by d have finite and positive measure. For any $x \in \mathcal{X}$ and $r \in (0, \infty)$, set the ball $B(x, r) \equiv \{y \in \mathcal{X} : d(x, y) < r\}$.

(i) The triple $(\mathcal{X}, d, \mu)$ is called a space of homogeneous type if there exists a constant $A_1 \in [1, \infty)$ such that for all $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$\mu(B(x, 2r)) \leq A_1 \mu(B(x, r)) \ (doubling \ property). \tag{1}$$

(ii) Let $\kappa \in (0, n]$. The triple $(\mathcal{X}, d, \mu)$ is called a (κ, n)-space if there exist constants $A_2 \in (0, 1]$ and $A_3 \in [1, \infty)$ such that for all $x \in \mathcal{X}$, $r \in (0, \operatorname{diam}(\mathcal{X})/2]$ and $\lambda \in [1, \operatorname{diam}(\mathcal{X})/(2r)]$,

$$A_2 \lambda^\kappa \mu(B(x, r)) \leq \mu(B(x, \lambda r)) \leq A_3 \lambda^n \mu(B(x, r)), \tag{2}$$

where $\operatorname{diam}(\mathcal{X}) \equiv \sup_{x, y \in \mathcal{X}} d(x, y)$.

A space of homogeneous type is called an RD-space, if it is a (κ, n)-space for some $\kappa \in (0, n]$, i.e., if some "reverse" doubling condition holds.

Obviously, a (κ, n)-space is a space of homogeneous type with $A_1 = A_3 2^n$. Conversely, a space of homogeneous type satisfies the second inequality of (2) with $A_3 = A_1$ and $n = \log_2 A_1$. Moreover, it was proved in [16, Remark 1] that $\mathcal{X}$ is an RD-space if and only if $\mathcal{X}$ is a space of homogeneous type with the additional property that there exists a constant $a_0 \in (1, \infty)$ such that for all $x \in \mathcal{X}$ and $r \in (0, \operatorname{diam}(\mathcal{X})/a_0)$, $B(x, a_0 r) \setminus B(x, r) \neq \emptyset$.

In what follows, we always set $V_r(x) \equiv \mu(B(x, r))$ and $V(x, y) \equiv \mu(B(x, d(x, y)))$ for all $x, y \in \mathcal{X}$ and $r \in (0, \infty)$.

Definition 2.2. ([31]) A positive function ρ on $\mathcal{X}$ is said to be admissible if there exist positive constants C_0 and k_0 such that for all $x, y \in \mathcal{X}$,

$$\frac{1}{\rho(x)} \leq C_0 \frac{1}{\rho(y)} \left(1 + \frac{d(x, y)}{\rho(y)}\right)^{k_0}. \tag{3}$$

We remark that the function ρ in Definition 2.2 does exist. Obviously, if ρ is a constant function, then ρ is admissible. Moreover, let $x_0 \in \mathcal{X}$ being fixed. The function $\rho(y) \equiv (1+d(x_0, y))^s$ for all $y \in \mathcal{X}$ with $s \in (-\infty, 1)$ also satisfies Definition 2.2 with $k_0 = s/(1-s)$ when $s \in [0, 1)$ and $k_0 = -s$ when $s \in (-\infty, 0)$. Another non-trivial class of admissible functions is given by the well-known reverse Hölder class $\mathcal{B}_q(\mathcal{X}, d, \mu)$, which is always written as $\mathcal{B}_q(\mathcal{X})$. Recall that a nonnegative potential V is said to be in $\mathcal{B}_q(\mathcal{X})$ with $q \in (1, \infty]$ if there exists a positive constant C such that for all balls B of $\mathcal{X}$,

$$\left(\frac{1}{\mu(B)} \int_B [V(y)]^q \, d\mu(y) \right)^{1/q} \le \frac{C}{\mu(B)} \int_B V(y) \, d\mu(y)$$

with the usual modification made when $q = \infty$. It is known that if $V \in \mathcal{B}_q(\mathcal{X})$ for certain $q \in (1, \infty]$, then V is an $A_\infty(\mathcal{X})$ weight in the sense of Muckenhoupt, and also $V \in \mathcal{B}_{q+\epsilon}(\mathcal{X})$ for some $\epsilon \in (0, \infty)$; see, for example, [27] and [28]. Thus $\mathcal{B}_q(\mathcal{X}) = \cup_{q_1 > q} \mathcal{B}_{q_1}(\mathcal{X})$. For all $V \in \mathcal{B}_q(\mathcal{X})$ with certain $q \in (1, \infty]$ and all $x \in \mathcal{X}$, set

$$\rho(x) \equiv [m(x, V)]^{-1} \equiv \sup \left\{ r > 0: \quad \frac{r^2}{\mu(B(x, r))} \int_{B(x, r)} V(y) \, d\mu(y) \le 1 \right\}; \qquad (4)$$

see, for example, [26] and also [31]. It was also proved in [31] that ρ in (4) is an admissible function if $n \ge 1$, $q > \max\{1, n/2\}$ and $V \in \mathcal{B}_q(\mathcal{X})$.

The following notion of approximations of the identity on RD-spaces was first introduced in [16], whose existence was given in Theorem 2.1 of [16].

Definition 2.3. Let $\epsilon_1 \in (0, 1]$ and $\epsilon_2 \in (0, \infty)$. A sequence $\{S_k\}_{k \in \mathbb{Z}}$ of bounded linear integral operators on $L^2(\mathcal{X})$ is said to be an approximation of the identity of order (ϵ_1, ϵ_2) (for short, (ϵ_1, ϵ_2)-AOTI), if there exists a positive constant A_4 such that for all $k \in \mathbb{Z}$ and all x, x', y and $y' \in \mathcal{X}$, $S_k(x, y)$, the integral kernel of S_k is a measurable function from $\mathcal{X} \times \mathcal{X}$ into $\mathbb{C}$ satisfying

(i) $|S_k(x, y)| \le A_4 \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y) + V(x, y)} \frac{2^{-k\epsilon_2}}{(2^{-k} + d(x, y))^{\epsilon_2}}$;

(ii) $|S_k(x, y) - S_k(x', y)| \le A_4 \left(\frac{d(x, x')}{2^{-k} + d(x, y)} \right)^{\epsilon_1} \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y) + V(x, y)} \frac{2^{-k\epsilon_2}}{(2^{-k} + d(x, y))^{\epsilon_2}}$

   for $d(x, x') \le (2^{-k} + d(x, y))/2$;

(iii) Property (ii) also holds with x and y interchanged;

(iv) $\int_{\mathcal{X}} S_k(x, z) \, d\mu(z) = 1 = \int_{\mathcal{X}} S_k(z, y) \, d\mu(z)$ for all $x, y \in \mathcal{X}$.

Remark 2.1. If a sequence $\{\widetilde{S}_t\}_{t > 0}$ of bounded linear integral operators on $L^2(\mathcal{X})$ satisfies (i) through (iv) of Definition 2.3 with 2^{-k} replaced by t, then we call $\{\widetilde{S}_t\}_{t > 0}$ a continuous approximation of the identity of order (ϵ_1, ϵ_2) (for short, continuous (ϵ_1, ϵ_2)-AOTI). For example, if $\{S_k\}_{k \in \mathbb{Z}}$ is an (ϵ_1, ϵ_2)-AOTI and if we set $\widetilde{S}_t(x, y) = S_k(x, y)$ for $t \in (2^{-k-1}, 2^{-k}]$ with $k \in \mathbb{Z}$, then $\{\widetilde{S}_t\}_{t > 0}$ is a continuous (ϵ_1, ϵ_2)-AOTI.

Definition 2.4. Let $\epsilon_1 \in (0, 1]$, $\epsilon_2 \in (0, \infty)$ and $\{S_t\}_{t > 0}$ be a continuous $(\epsilon_1, \epsilon_2) - $ AOTI. Let ρ be admissible.

(i) For any $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ and $x \in \mathcal{X}$, the radial maximal function $S^+(f)$ is defined by

$$S^+(f)(x) \equiv \sup_{t>0} |S_t(f)(x)|;$$

(ii) For any $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ and $x \in \mathcal{X}$, the radial maximal function $S^+_\rho(f)$ associated to ρ is defined by

$$S^+_\rho(f)(x) \equiv \sup_{0<t<\rho(x)} |S_t(f)(x)|.$$

3. **Localized BMO and BLO spaces.** This section is divided into two subsections. In Section 3.1, we introduce a localized BMO-type space $\mathrm{BMO}_\rho(\mathcal{X})$ and establish its several equivalent characterizations, John-Nirenberg inequality and some other properties; while Section 3.2 is devoted to the study of a corresponding localized BLO-type space $\mathrm{BLO}_\rho(\mathcal{X})$.

3.1. **A localized BMO space.**

Definition 3.1. Let ρ be an admissible function on $\mathcal{X}$, $\mathcal{D} \equiv \{B(x,r) \subset \mathcal{X} : x \in \mathcal{X}, r \geq \rho(x)\}$ and $q \in [1, \infty)$.

(i) A function $f \in L^q_{\mathrm{loc}}(\mathcal{X})$ is said to be in the space $\mathrm{BMO}^q(\mathcal{X})$ if

$$\|f\|_{\mathrm{BMO}^q(\mathcal{X})} \equiv \sup_{B \subset \mathcal{X}} \left\{ \frac{1}{\mu(B)} \int_B |f(y) - f_B|^q d\mu(y) \right\}^{1/q} < \infty.$$

(ii) A function $f \in L^q_{\mathrm{loc}}(\mathcal{X})$ is said to be in the space $\mathrm{BMO}^q_\rho(\mathcal{X})$ if

$$\|f\|_{\mathrm{BMO}^q_\rho(\mathcal{X})} \equiv \sup_{B \notin \mathcal{D}} \left\{ \frac{1}{\mu(B)} \int_B |f(y) - f_B|^q d\mu(y) \right\}^{1/q}$$
$$+ \sup_{B \in \mathcal{D}} \left\{ \frac{1}{\mu(B)} \int_B |f(y)|^q d\mu(y) \right\}^{1/q} < \infty.$$

Remark 3.1. (i) The space $\mathrm{BMO}^q(\mathcal{X})$ with $q \in [1, \infty)$ coincides with $\mathrm{BMO}^1(\mathcal{X})$; see [3]. We denote $\mathrm{BMO}^1(\mathcal{X})$ simply by $\mathrm{BMO}(\mathcal{X})$.

(ii) We also denote $\mathrm{BMO}^1_\rho(\mathcal{X})$ simply by $\mathrm{BMO}_\rho(\mathcal{X})$. The localized space $\mathrm{BMO}_\rho(\mathbb{R}^d)$ when $\rho \equiv 1$ was first introduced by Goldberg [12]. If $q > \frac{d}{2}$, $V \in \mathcal{B}_q(\mathbb{R}^d)$ and ρ is as in (4), then $\mathrm{BMO}_\rho(\mathbb{R}^d)$ is just the space $\mathrm{BMO}_\mathcal{L}(\mathbb{R}^d)$ introduced by Dziubánski et al in [9]. For all $q \in [1, \infty)$, $\mathrm{BMO}^q_\rho(\mathcal{X}) \subsetneq \mathrm{BMO}(\mathcal{X})$.

(iii) Let $q \in [1, \infty)$, $a \in (0, \infty)$ and $\mathcal{D}_a \equiv \{B(x,r) \subset \mathcal{X} : r \geq a\}$. Define the space $\mathrm{BMO}^q_a(\mathcal{X})$ as in Definition 3.1 (ii) with $\mathcal{D}$ replaced by $\mathcal{D}_a$. Then, (1) implies that for all $q \in [1, \infty)$ and fixed $a_1, a_2 \in (0, \infty)$, $\mathrm{BMO}^q_{a_1}(\mathcal{X}) = \mathrm{BMO}^q_{a_2}(\mathcal{X})$ with equivalent norms. From this, it further follows that if $\mu(\mathcal{X}) < \infty$, then for all $q \in [1, \infty)$ and any fixed $a \in (0, \infty)$, $\mathrm{BMO}^q_\rho(\mathcal{X}) = \mathrm{BMO}^q_a(\mathcal{X})$ with equivalent norms. In fact, by (2), there exists a positive constant M such that for all $x, y \in \mathcal{X}$, $d(x,y) \leq M$. This together with Lemma 2.1 in [31] implies that there exist positive constants C and $\widetilde{C}$ such that for all $x \in \mathcal{X}$, $C \leq \rho(x) \leq \widetilde{C}$. Thus, for all $q \in [1, \infty)$, $\mathrm{BMO}^q_C(\mathcal{X}) \subset \mathrm{BMO}^q_\rho(\mathcal{X}) \subset \mathrm{BMO}^q_{\widetilde{C}}(\mathcal{X})$ which implies the desired conclusion.

The following result follows from Definition 3.1.

Lemma 3.1. *Let ρ be an admissible function on $\mathcal{X}$ and $q \in [1, \infty)$. Then $\mathrm{BMO}_\rho(\mathcal{X}) = \mathrm{BMO}_\rho^q(\mathcal{X})$ with equivalent norms.*

Proof. Assume that $f \in \mathrm{BMO}_\rho^q(\mathcal{X})$. Then by the Hölder inequality, $f \in \mathrm{BMO}_\rho(\mathcal{X})$ and $\|f\|_{\mathrm{BMO}_\rho(\mathcal{X})} \leq \|f\|_{\mathrm{BMO}_\rho^q(\mathcal{X})}$. Conversely, if $f \in \mathrm{BMO}_\rho(\mathcal{X})$, then by Remark 3.1 (i) and Definition 3.1,

$$\|f\|_{\mathrm{BMO}_\rho^q(\mathcal{X})} \lesssim \|f\|_{\mathrm{BMO}^q(\mathcal{X})} + \sup_{B \in \mathcal{D}} |f_B| \lesssim \|f\|_{\mathrm{BMO}_\rho(\mathcal{X})},$$

which implies that $f \in \mathrm{BMO}_\rho^q(\mathcal{X})$ and $\|f\|_{\mathrm{BMO}_\rho^q(\mathcal{X})} \lesssim \|f\|_{\mathrm{BMO}_\rho(\mathcal{X})}$. Thus $\mathrm{BMO}_\rho(\mathcal{X}) = \mathrm{BMO}_\rho^q(\mathcal{X})$ with equivalent norms, which completes the proof of Lemma 3.1. $\square$

Recall that the classical John-Nirenberg inequality (see [3]) says that there exist positive constants C_1 and C_2 such that for all $f \in \mathrm{BMO}(\mathcal{X})$, balls B and $\lambda > 0$,

$$\mu\left(\{x \in B : |f(x) - f_B| > \lambda\}\right) \leq C_1 \mu(B) \exp\left\{-\frac{C_2 \lambda}{\|f\|_{\mathrm{BMO}(\mathcal{X})}}\right\}. \tag{5}$$

From this, we deduce a variant of the John-Nirenberg inequality suitable for $\mathrm{BMO}_\rho(\mathcal{X})$ as follows.

Theorem 3.1. *Let ρ be an admissible function on $\mathcal{X}$ and $\mathcal{D}$ be as in Definition 3.1. If $f \in \mathrm{BMO}_\rho(\mathcal{X})$, then there exist positive constants C_3 and C_4 such that for all balls B and $\lambda > 0$,*

$$\mu\left(\{x \in B : |f(x) - f_B| > \lambda\}\right) \leq C_3 \mu(B) \exp\left\{-\frac{C_4 \lambda}{\|f\|_{\mathrm{BMO}_\rho(\mathcal{X})}}\right\}, \tag{6}$$

and, moreover, for all $B \in \mathcal{D}$,

$$\mu\left(\{x \in B : |f(x)| > \lambda\}\right) \leq C_3 \mu(B) \exp\left\{-\frac{C_4 \lambda}{\|f\|_{\mathrm{BMO}_\rho(\mathcal{X})}}\right\}. \tag{7}$$

Proof. The inequality (6) follows from (5) and Definition 3.1 directly. To show (7), let $B \in \mathcal{D}$. If $\lambda > 2\|f\|_{\mathrm{BMO}_\rho(\mathcal{X})}$, by the definition, we have $\lambda > 2|f|_B$. Thus for all balls B in $\mathcal{D}$, we obtain

$$\mu\left(\{x \in B : |f(x)| > \lambda\}\right) \leq \mu\left(\{x \in B : |f(x) - f_B| > \lambda/2\}\right),$$

which together (6) yields (7); if $0 < \lambda \leq 2\|f\|_{\mathrm{BMO}_\rho(\mathcal{X})}$, we then have

$$\mu\left(\{x \in B : |f(x)| > \lambda\}\right) \leq \mu(B) \lesssim \mu(B) \exp\left\{-\frac{C_2 \lambda}{\|f\|_{\mathrm{BMO}_\rho(\mathcal{X})}}\right\}.$$

This finishes the proof of Theorem 3.1. $\square$

Remark 3.2. Let ρ be an admissible function on $\mathcal{X}$ and $q \in [1, \infty)$. Applying Theorem 3.1, we can also obtain that $\mathrm{BMO}_\rho(\mathcal{X}) = \mathrm{BMO}_\rho^q(\mathcal{X})$ with equivalent norms.

We now establish the relation between $\mathrm{BMO}(\mathcal{X})$ and $\mathrm{BMO}_\rho(\mathcal{X})$ in terms of certain approximation of the identity. To begin with, let ρ be an admissible function on $\mathcal{X}$. In [31], it was proved that there exist a nonnegative function K_ρ on $\mathcal{X} \times \mathcal{X}$ and a positive constant C_5 such that

$(\mathrm{K})_1$ $K_\rho(x, y) = 0$ if $x, y \in \mathcal{X}$ satisfying $d(x, y) > C_5 \min\{\rho(x), \rho(y)\}$;
$(\mathrm{K})_2$ $K_\rho(x, y) \le C_5 \frac{1}{V_{\rho(x)}(x) + V_{\rho(y)}(y)}$ for all $x, y \in \mathcal{X}$;
$(\mathrm{K})_3$ $K_\rho(x, y) = K_\rho(y, x)$ for all $x, y \in \mathcal{X}$;
$(\mathrm{K})_4$ $\int_{\mathcal{X}} K_\rho(x, y)\, d\mu(x) = 1$ for all $y \in \mathcal{X}$.
For all $x \in \mathcal{X}$, let

$$K_\rho(f)(x) \equiv \int_{\mathcal{X}} K_\rho(x, y) f(y)\, d\mu(y). \tag{8}$$

It was proved in [31] that if $f \in H_\rho^1(\mathcal{X})$, the Hardy space associated to ρ, then $f - K_\rho(f) \in H^1(\mathcal{X})$, where $H^1(\mathcal{X})$ is the Hardy space studied in [15, 16, 14], which coincides with the atomic Hardy space $H_{at}^1(\mathcal{X})$ of Coifman and Weiss in [3]. Moreover, there exists a positive constant C such that for all $f \in H_\rho^1(\mathcal{X})$,

$$\|f - K_\rho(f)\|_{H^1(\mathcal{X})} \le C\|f\|_{H_\rho^1(\mathcal{X})}. \tag{9}$$

On the other hand, it was showed in [32] that the dual space of $H_\rho^1(\mathcal{X})$ is $\mathrm{BMO}_\rho(\mathcal{X})$. From these facts, we deduce the following corollary.

Corollary 3.1. *Let ρ be an admissible function on $\mathcal{X}$ and K_ρ be as in* (8). *Then*
  (i) $\mathrm{BMO}_\rho(\mathcal{X}) = \{b \in \mathrm{BMO}(\mathcal{X}) : K_\rho(b) \in L^\infty(\mathcal{X})\}$; *moreover, for all $b \in \mathrm{BMO}_\rho(\mathcal{X})$,*
$\|b\|_{\mathrm{BMO}_\rho(\mathcal{X})} \sim \|K_\rho(b)\|_{L^\infty(\mathcal{X})} + \|b\|_{\mathrm{BMO}(\mathcal{X})}$.
  (ii) If $f \in \mathrm{BMO}(\mathcal{X})$, then $f - K_\rho(f) \in \mathrm{BMO}_\rho(\mathcal{X})$; moreover, there exists a positive constant C such that for all $f \in \mathrm{BMO}(\mathcal{X})$,

$$\|f - K_\rho(f)\|_{\mathrm{BMO}_\rho(\mathcal{X})} \le C\|f\|_{\mathrm{BMO}(\mathcal{X})}.$$

Proof. We first prove (i). Assume that $b \in \mathrm{BMO}(\mathcal{X})$ with $K_\rho(b) \in L^\infty(\mathcal{X})$. Recall that $H_\rho^1(\mathcal{X}) \subset L^1(\mathcal{X})$ (see Lemma 3.1 in [31]). For any $f \in H_\rho^1(\mathcal{X})$, by $(\mathrm{K})_3$, (9) and $\left(H^1(\mathcal{X})\right)^* = \mathrm{BMO}(\mathcal{X})$ (see [3]), we have

$$\left| \int_{\mathcal{X}} b(x) f(x)\, d\mu(x) \right| \le \left| \int_{\mathcal{X}} b(x) \left[f(x) - K_\rho(f)(x) \right] d\mu(x) \right| + \left| \int_{\mathcal{X}} b(x) K_\rho(f)(x)\, d\mu(x) \right|$$

$$\le \|b\|_{\mathrm{BMO}(\mathcal{X})} \|f - K_\rho(f)\|_{H^1(\mathcal{X})} + \left| \int_{\mathcal{X}} f(x) K_\rho(b)(x)\, d\mu(x) \right|$$

$$\lesssim \|f\|_{H_\rho^1(\mathcal{X})} \left[\|b\|_{\mathrm{BMO}(\mathcal{X})} + \|K_\rho(b)\|_{L^\infty(\mathcal{X})} \right].$$

Thus by $(H_\rho^1(\mathcal{X}))^* = \mathrm{BMO}_\rho(\mathcal{X})$, we obtain $b \in \mathrm{BMO}_\rho(\mathcal{X})$ and

$$\|b\|_{\mathrm{BMO}_\rho(\mathcal{X})} \lesssim \|b\|_{\mathrm{BMO}(\mathcal{X})} + \|K_\rho(b)\|_{L^\infty(\mathcal{X})}.$$

Conversely, assume that $b \in \mathrm{BMO}_\rho(\mathcal{X})$. By $(K)_1$ and $(K)_2$, for all $x \in \mathcal{X}$, we have

$$|K_\rho(b)(x)| \leq \int_{d(x,y) \leq C_5 \rho(x)} |K_\rho(x,y)b(y)| \, d\mu(y)$$
$$\lesssim \frac{1}{V_{C_5\rho(x)}(x)} \int_{B(x, C_5\rho(x))} |b(y)| \, d\mu(y) \lesssim \|b\|_{\mathrm{BMO}_\rho(\mathcal{X})}.$$

This shows (i).

To see (ii), by $(K)_3$, (9) and $(H^1(\mathcal{X}))^* = \mathrm{BMO}(\mathcal{X})$, we have that for all $f \in \mathrm{BMO}(\mathcal{X})$ and $b \in H^1_\rho(\mathcal{X})$,

$$\left| \int_{\mathcal{X}} [f(x) - K_\rho(f)(x)]b(x) \, d\mu(x) \right| = \left| \int_{\mathcal{X}} f(x)[b(x) - K_\rho(b)(x)] \, d\mu(x) \right|$$
$$\lesssim \|f\|_{\mathrm{BMO}(\mathcal{X})} \|b\|_{H^1_\rho(\mathcal{X})},$$

which together with $(H^1_\rho(\mathcal{X}))^* = \mathrm{BMO}_\rho(\mathcal{X})$ implies that $f - K_\rho(f) \in \mathrm{BMO}_\rho(\mathcal{X})$ and the desired estimate. This finishes the proof of Corollary 3.1. $\square$

3.2. A localized BLO space.

Definition 3.2. Let ρ and $\mathcal{D}$ be as in Definition 3.1 and $q \in [1, \infty)$.

(i) A function $f \in L^q_{\mathrm{loc}}(\mathcal{X})$ is said to be in the space $\mathrm{BLO}^q(\mathcal{X})$ if

$$\|f\|_{\mathrm{BLO}^q(\mathcal{X})} \equiv \sup_{B \subset \mathcal{X}} \left\{ \frac{1}{\mu(B)} \int_B \left[f(y) - \operatorname*{essinf}_B f \right]^q \, d\mu(y) \right\}^{1/q} < \infty.$$

(ii) A function $f \in L^q_{\mathrm{loc}}(\mathcal{X})$ is said to be in the space $\mathrm{BLO}_\rho(\mathcal{X})$ if

$$\|f\|_{\mathrm{BLO}^q_\rho(\mathcal{X})} \equiv \sup_{B \notin \mathcal{D}} \left\{ \frac{1}{\mu(B)} \int_B \left[f(y) - \operatorname*{essinf}_B f \right]^q \, d\mu(y) \right\}^{1/q}$$
$$+ \sup_{B \in \mathcal{D}} \left\{ \frac{1}{\mu(B)} \int_B |f(y)|^q \, d\mu(y) \right\}^{1/q} < \infty.$$

Remark 3.3. (i) The space $\mathrm{BLO}^1(\mathbb{R}^d)$ with the Lebesgue measure was introduced by Coifman and Rochberg [1], and extended by Jiang [19] to the setting of $\mathbb{R}^d$ with a non-doubling measure. Let $q \in [1, \infty)$. Then the facts that $\mathrm{BLO}^1(\mathcal{X}) \subset \mathrm{BMO}(\mathcal{X}) = \mathrm{BMO}^q(\mathcal{X})$ together with the Hölder inequality imply that $\mathrm{BLO}^q(\mathcal{X}) = \mathrm{BLO}^1(\mathcal{X})$ with equivalent norms. We denote $\mathrm{BLO}^1(\mathcal{X})$ simply by $\mathrm{BLO}(\mathcal{X})$. Notice that $\mathrm{BLO}(\mathcal{X})$ is not a linear space.

(ii) We also denote $\mathrm{BLO}^1_\rho(\mathcal{X})$ simply by $\mathrm{BLO}_\rho(\mathcal{X})$. The localized BLO space was first introduced in [18] in the setting of $\mathbb{R}^d$ with a non-doubling measure. For all $q \in [1, \infty)$, $\mathrm{BLO}^q_\rho(\mathcal{X}) \subset \mathrm{BMO}^q_\rho(\mathcal{X})$. Even when $\rho \equiv 1$, it is not so difficult to show that for all $q \in [1, \infty)$, $\mathrm{BLO}^q_\rho(\mathbb{R}^d)$ is a proper subspace of $\mathrm{BMO}^q_\rho(\mathbb{R}^d)$.

(iii) Let $q \in [1, \infty)$, $a \in (0, \infty)$ and $\mathcal{D}_a \equiv \{B(x, r) \subset \mathcal{X} : r \geq a\}$. Define the space $\mathrm{BLO}^q_a(\mathcal{X})$ as in Definition 3.2 (ii) with $\mathcal{D}$ replaced by $\mathcal{D}_a$. If $\mu(\mathcal{X}) < \infty$, then for all $q \in [1, \infty)$ and admissible functions ρ, and any fixed $a \in (0, \infty)$, $\mathrm{BLO}^q_\rho(\mathcal{X}) = \mathrm{BLO}^q_a(\mathcal{X})$ with equivalent norms. The proof is similar to that of Remark 3.1 (iii) and is omitted.

The following result follows from Definitions 3.1 and 3.2, whose proof is similar to that of Lemma 3.1 and is omitted.

Lemma 3.2. *Let ρ be an admissible function on $\mathcal{X}$ and $q \in [1, \infty)$. Then $\mathrm{BLO}_\rho(\mathcal{X}) = \mathrm{BLO}_\rho^q(\mathcal{X})$ with equivalent norms.*

Theorem 3.2. *Let ρ be an admissible function on $\mathcal{X}$. There exists a positive constant C such that for all $f \in \mathrm{BLO}_\rho(\mathcal{X})$, $f(x) \geq -C\|f\|_{\mathrm{BLO}_\rho(\mathcal{X})}$ for almost all $x \in \mathcal{X}$. Moreover, the following statements are equivalent:*
(i) $f \in \mathrm{BLO}_\rho(\mathcal{X})$;
(ii) $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ and there exists a nonnegative constant A such that

$$\sup_{B \notin \mathcal{D}} \left\{ \frac{1}{\mu(B)} \int_B \left[f(y) - \operatorname*{essinf}_B f \right] d\mu(y) \right\}$$
$$+ \sup_{B \in \mathcal{D}} \left\{ \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y) + \left| \operatorname*{essinf}_B f \right| \right\} \leq A;$$

(iii) $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ and there exists a nonnegative constant $\widetilde{C}$ such that

$$\sup_{B \subset \mathcal{X}} \left\{ \frac{1}{\mu(B)} \int_B \left[f(y) - \operatorname*{essinf}_B f \right] d\mu(y) \right\} + \sup_{B \in \mathcal{D}} \left\{ \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y) \right\} \leq \widetilde{C}.$$

Moreover, $\|f\|_{\mathrm{BLO}_\rho(\mathcal{X})}$, $\inf\{A\}$ and $\inf\{\widetilde{C}\}$ are mutually equivalent.

Proof. Let $f \in \mathrm{BLO}_\rho(\mathcal{X})$. For all balls $B \equiv B(x_0, \rho(x_0)/2)$, we have that

$$\operatorname*{essinf}_B f \geq -\frac{1}{\mu(B)} \int_B \left[f(x) - \operatorname*{essinf}_B f \right] d\mu(x) - \frac{1}{\mu(B)} \int_B |f(x)| \, d\mu(x) \gtrsim -\|f\|_{\mathrm{BLO}_\rho(\mathcal{X})},$$

which together with $\mathcal{X} = \cup_x B(x, \rho(x)/2)$ and the Vitali-Wiener type covering lemma (see [3, p. 623]) implies that there exists certain positive constant C such that for μ-a. e. $x \in \mathcal{X}$, $f(x) \geq -C\|f\|_{\mathrm{BLO}_\rho(\mathcal{X})}$. From this, it is easy to see that (i) implies (ii). Obviously, (ii) implies (iii) and (iii) implies (i). Thus we complete the proof of Theorem 3.2. $\qquad\square$

Remark 3.4. (i) From Theorem 3.2 (ii) and Definition 3.2 (i), it follows that $\mathrm{BLO}_\rho(\mathcal{X}) \subset \mathrm{BLO}(\mathcal{X})$.

(ii) During this paper being written, we learnt that when $V \in B_q(\mathbb{R}^d)$ with $q > d/2$, and ρ is as in (4), Theorem 3.2 (iii) was used, independently, by Gao, Jiang and Tang [11] to introduce the space $\mathrm{BLO}_\mathcal{L}(\mathbb{R}^d)$ corresponding to the Schrödinger operator $\mathcal{L} = -\Delta + V$.

As a consequence of Theorem 3.2, we have the following corollary.

Corollary 3.2. *Let ρ be an admissible function on $\mathcal{X}$. Then $\mathrm{BLO}_\rho(\mathcal{X}) = \mathrm{BMO}_\rho(\mathcal{X}) \cap \mathrm{BLO}(\mathcal{X})$. Moreover, for all $f \in \mathrm{BLO}_\rho(\mathcal{X})$,*

$$\|f\|_{\mathrm{BLO}_\rho(\mathcal{X})} \sim \|f\|_{\mathrm{BMO}_\rho(\mathcal{X})} + \|f\|_{\mathrm{BLO}(\mathcal{X})}.$$

Proof. Let $f \in \mathrm{BMO}_\rho(\mathcal{X}) \cap \mathrm{BLO}(\mathcal{X})$ first. By Definitions 3.1 and 3.2, $f \in \mathrm{BLO}_\rho(\mathcal{X})$ and $\|f\|_{\mathrm{BLO}_\rho(\mathcal{X})} \leq \|f\|_{\mathrm{BMO}_\rho(\mathcal{X})} + \|f\|_{\mathrm{BLO}(\mathcal{X})}$. Conversely, assume that $f \in \mathrm{BLO}_\rho(\mathcal{X})$. It follows from Definitions 3.1 and 3.2 that $f \in \mathrm{BMO}_\rho(\mathcal{X})$ and $\|f\|_{\mathrm{BMO}_\rho(\mathcal{X})} \lesssim \|f\|_{\mathrm{BLO}_\rho(\mathcal{X})}$, which together with Remark 3.4 (i) completes the proof of Corollary 6. $\square$

As a consequence of Corollary 3.1 and Corollary 3.2, we have the following result.

Corollary 3.3. *Let ρ be an admissible function on $\mathcal{X}$ and K_ρ be as in* (8). *Then*

$$\mathrm{BLO}_\rho(\mathcal{X}) = \{f \in \mathrm{BLO}(\mathcal{X}): \ K_\rho(f) \in L^\infty(\mathcal{X})\}.$$

Proof. Assume that $f \in \mathrm{BLO}_\rho(\mathcal{X})$ first. Then by Corollary 3.2, $f \in \mathrm{BLO}(\mathcal{X}) \cap \mathrm{BMO}_\rho(\mathcal{X})$. From this and Corollary 3.1 (i), it follows that $K_\rho(f) \in L^\infty(\mathcal{X})$. Conversely, if $f \in \mathrm{BLO}(\mathcal{X})$ and $K_\rho(f) \in L^\infty(\mathcal{X})$, then the obvious fact $\mathrm{BLO}(\mathcal{X}) \subset \mathrm{BMO}(\mathcal{X})$ together with another application of Corollary 3.1 (i) implies that $f \in \mathrm{BMO}_\rho(\mathcal{X})$, which together with Corollary 3.2 yields that $f \in \mathrm{BLO}_\rho(\mathcal{X})$. This finishes the proof of Corollary 3.3. $\square$

Theorem 3.3. *Let ρ be an admissible function on $\mathcal{X}$ and K_ρ be as in* (8). *Then there exists a positive constant C such that for all $f \in \mathrm{BLO}(\mathcal{X})$, $f - K_\rho f \in \mathrm{BLO}_\rho(\mathcal{X})$ and*

$$\|f - K_\rho(f)\|_{\mathrm{BLO}_\rho(\mathcal{X})} \leq C\|f\|_{\mathrm{BLO}(\mathcal{X})}.$$

Proof. Let $f \in \mathrm{BLO}(\mathcal{X})$. By the homogeneity of $\|\cdot\|_{\mathrm{BLO}(\mathcal{X})}$ and $\|\cdot\|_{\mathrm{BLO}_\rho(\mathcal{X})}$, we may assume that $\|f\|_{\mathrm{BLO}(\mathcal{X})} = 1$. Let $B \equiv B(x_0, r) \in \mathcal{D}$. Observe that by (3), for any $a \in (0, \infty)$, there exists a constant $\widetilde{C}_a \in [1, \infty)$ such that for all $x, y \in \mathcal{X}$ with $d(x, y) \leq a\rho(x)$,

$$\rho(y)/\widetilde{C}_a \leq \rho(x) \leq \widetilde{C}_a\rho(y). \tag{10}$$

By this and $r \geq \rho(x_0)$, we obtain that for all $x \in B$, $\rho(x) \lesssim r$. Then there exists a positive constant C such that for all $x \in B$, $B(x, C_5\rho(x)) \subset CB$. By $(\mathrm{K})_1$ through $(\mathrm{K})_4$, (1) and the Tonelli theorem, we obtain

$$\frac{1}{\mu(B)} \int_B |f(x) - K_\rho(f)(x)| \, d\mu(x)$$
$$\leq \frac{1}{\mu(B)} \int_B \left\{ \left[f(x) - \underset{CB}{\mathrm{essinf}} \, f \right] + \left| K_\rho \left(f - \underset{CB}{\mathrm{essinf}} \, f \right)(x) \right| \right\} d\mu(x)$$
$$\lesssim 1 + \frac{1}{\mu(B)} \int_{CB} \int_{B(y, C_5\rho(y))} \frac{|f(y) - \mathrm{essinf}\,_{CB} f|}{V_{\rho(x)}(x) + V_{\rho(y)}(y)} \, d\mu(x) \, d\mu(y) \lesssim 1. \tag{11}$$

On the other hand, let $B \equiv B(x_0, r) \notin \mathcal{D}$. Using $r < \rho(x_0)$ and (10) with $a = 1$, we obtain that there exists a constant $\widetilde{A}_1 \in [1, \infty)$ such that for all $x, y \in B$, $B(x, \rho(x)) \subset B(y, \widetilde{A}_1\rho(y))$. From this together with (1), it follows that for all $x, y \in B$,

$$\left| \underset{B(y, \rho(y))}{\mathrm{essinf}} \, f - \underset{B(x, \rho(x))}{\mathrm{essinf}} \, f \right| \lesssim 1.$$

By this together with $(K)_1$ through $(K)_4$ and (1), we have that for all x, $y \in B$,

$$\left| K_\rho(f)(y) - \operatorname*{essinf}_{B(x,\,\rho(x))} f \right| \leq \left| K_\rho(f)(y) - \operatorname*{essinf}_{B(y,\,\rho(y))} f \right| + \left| \operatorname*{essinf}_{B(y,\,\rho(y))} f - \operatorname*{essinf}_{B(x,\,\rho(x))} f \right| \lesssim 1.$$

From this fact, we deduce that

$$[f - K_\rho(f)]_B - \operatorname*{essinf}_B [f - K_\rho(f)]$$

$$\leq \frac{1}{\mu(B)} \int_B \left\{ [f(x) - K_\rho(f)(x)] - \operatorname*{essinf}_B f - \operatorname*{essinf}_B [-K_\rho(f)] \right\} d\mu(x)$$

$$\lesssim 1 + \frac{1}{\mu(B)} \int_B \left\{ \left[-K_\rho(f)(x) + \operatorname*{essinf}_{B(x,\,\rho(x))} f \right] + \left[- \operatorname*{essinf}_{B(x,\,\rho(x))} f - \operatorname*{essinf}_B [-K_\rho(f)] \right] \right\} d\mu(x)$$

$$\lesssim 1.$$

This together with (11) gives the desired estimate and hence, finishes the proof of Theorem 3.3. $\qquad \square$

4. **Boundedness of the natural and the Hardy-Littlewood maximal functions.** In this section, we first obtain the boundedness of the natural maximal function, the Hardy-Littlewood maximal function and their localized versions from $\mathrm{BMO}_\rho(\mathcal{X})$ to $\mathrm{BLO}_\rho(\mathcal{X})$; as an application, we then establish several equivalent characterizations for $\mathrm{BLO}_\rho(\mathcal{X})$ via the localized natural maximal function.

Definition 4.1. Let ρ be an admissible function on $\mathcal{X}$. For all $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ and $x \in \mathcal{X}$, define $\mathrm{HL}(f)(x) \equiv \sup_{x \in B}\{|f|_B\}$, $M(f)(x) \equiv \sup_{x \in B}\{f_B\}$, $\mathrm{HL}_\rho(x) \equiv \sup_{x \in B,\, B \notin \mathcal{D}}\{|f|_B\}$, and $M_\rho(f)(x) \equiv \sup_{x \in B,\, B \notin \mathcal{D}}\{f_B\}$.

Theorem 4.1. *Let ρ be an admissible function on $\mathcal{X}$. Then M_ρ is bounded from $\mathrm{BMO}_\rho(\mathcal{X})$ to $\mathrm{BLO}_\rho(\mathcal{X})$, namely, there exists a positive constant C such that for all $f \in \mathrm{BMO}_\rho(\mathcal{X})$, $M_\rho(f) \in \mathrm{BLO}_\rho(\mathcal{X})$ and*

$$\|M_\rho(f)\|_{\mathrm{BLO}_\rho(\mathcal{X})} \leq C\|f\|_{\mathrm{BMO}_\rho(\mathcal{X})}.$$

Proof. Let $f \in \mathrm{BMO}_\rho(\mathcal{X})$. By the homogeneity of $\|\cdot\|_{\mathrm{BMO}_\rho(\mathcal{X})}$ and $\|\cdot\|_{\mathrm{BLO}_\rho(\mathcal{X})}$, we may assume that $\|f\|_{\mathrm{BMO}_\rho(\mathcal{X})} = 1$. We first prove that for all balls $B \equiv B(x_0, r) \in \mathcal{D}$,

$$[\mathrm{HL}_\rho(f)]_B \lesssim 1. \tag{12}$$

From this, it follows that for all balls $B \in \mathcal{D}$, $(|M_\rho(f)|)_B \lesssim 1$, which implies that $|M_\rho(f)(x)| < \infty$ for μ-a.e. $x \in \mathcal{X}$.

To prove (12), for all balls $B \in \mathcal{D}$, write

$$[\mathrm{HL}_\rho(f)]_B \leq [\mathrm{HL}_\rho(f\chi_{3B})]_B + [\mathrm{HL}_\rho(f\chi_{(3B)^\complement})]_B.$$

The Hölder inequality together with the $L^2(\mathcal{X})$-boundedness of HL (see [2]) and Lemma 3.1 gives us that

$$[\mathrm{HL}_\rho(f\chi_{3B})]_B \leq \{([\mathrm{HL}(f\chi_{3B})]^2)_B\}^{1/2} \lesssim \{(|f\chi_{3B}|^2)_B\}^{1/2} \lesssim 1. \tag{13}$$

We now claim that for all $y \in B$, $\mathrm{HL}_\rho(f\chi_{(3B)^\complement})(y) \lesssim 1$. In fact, for all balls $\widetilde{B} \equiv B(\widetilde{x}, \widetilde{r}) \notin \mathcal{D}$ such that $\widetilde{B} \cap B \neq \emptyset$, we have that either $\widetilde{B} \subset 3B$ or $B \subset 3\widetilde{B}$. If $\widetilde{B} \subset 3B$, then $(|f\chi_{(3B)^\complement}|)_{\widetilde{B}} = 0$. If $B \subset 3\widetilde{B}$, then by (10), there exists a constant $\widetilde{C} \in [1, \infty)$ such that $\widetilde{C}\widetilde{r} \geq \rho(\widetilde{x})$. Thus, using (1), we have that $(|f\chi_{(3B)^\complement}|)_{\widetilde{B}} \lesssim (|f|)_{\widetilde{C}\widetilde{B}} \lesssim 1$. The claim then follows from the two estimates above, which together with (13) leads to (12).

We now prove that there exists a positive constant C such that for all balls $B \equiv B(x_0, r) \notin \mathcal{D}$,

$$[M_\rho(f)]_B \leq C + \operatorname*{essinf}_B M_\rho(f). \tag{14}$$

Write

$$[M_\rho(f)]_B \leq \{M_\rho[(f - f_B)\chi_{3B}]\}_B + \left\{M_\rho\left[f_B\chi_{3B} + f\chi_{(3B)^\complement}\right]\right\}_B.$$

Using the Hölder inequality, the $L^2(\mathcal{X})$-boundedness of HL, (1) and Lemma 3.1, we obtain that

$$\{M_\rho[(f - f_B)\chi_{3B}]\}_B \leq \{\mathrm{HL}[(f - f_B)\chi_{3B}]\}_B \lesssim [(|f - f_{3B}|^2)_{3B}]^{1/2} \lesssim 1. \tag{15}$$

Now we show that for all $x, y \in B$, $M_\rho(f_B\chi_{3B} + f\chi_{(3B)^\complement})(x) \leq \widetilde{C} + M_\rho(f)(y)$. For all balls $\widetilde{B} \equiv B(\widetilde{x}, \widetilde{r}) \notin \mathcal{D}$ containing x, we have either $\widetilde{B} \subset 3B$ or $B \subset 3\widetilde{B}$. Assume that $\widetilde{B} \subset 3B$ first. Then

$$\left[f_B\chi_{3B} + f\chi_{(3B)^\complement}\right]_{\widetilde{B}} = f_B \leq M_\rho(f)(y). \tag{16}$$

Now we assume that $B \subset 3\widetilde{B}$. If $3\widetilde{B} \notin \mathcal{D}$, then the fact $y \in B \subset 3\widetilde{B}$ gives us that $f_{3\widetilde{B}} \leq M_\rho(f)(y)$, which together with (1) implies that

$$\left[f_B\chi_{3B} + f\chi_{(3B)^\complement}\right]_{\widetilde{B}} - M_\rho(f)(y)$$
$$= \left[\left(f_B - f_{3\widetilde{B}}\right)\chi_{3B} + \left(f - f_{3\widetilde{B}}\right)\chi_{(3B)^\complement}\right]_{\widetilde{B}} + f_{3\widetilde{B}} - M_\rho(f)(y)$$
$$\leq \left[|f_B - f_{3\widetilde{B}}|\chi_{3B}\right]_{\widetilde{B}} + \left(|f - f_{3\widetilde{B}}|\right)_{\widetilde{B}} \lesssim 1.$$

If $3\widetilde{B} \in \mathcal{D}$, then $\rho(\widetilde{x}) \leq 3\widetilde{r}$. Since $B, \widetilde{B} \notin \mathcal{D}$, $x, y \in B$ and $\widetilde{x}, x \in \widetilde{B}$, by (10), $\rho(y) \sim \rho(x_0) \sim \rho(x) \sim \rho(\widetilde{x}) > \widetilde{r}$, which implies that $\rho(y) \sim \rho(\widetilde{x}) \sim \widetilde{r}$. Let $\widetilde{A} \in [1, \infty)$ satisfying $\rho(y) > \widetilde{r}/\widetilde{A}$. Then $B(y, \widetilde{r}/\widetilde{A}) \notin \mathcal{D}$. By the fact that $f_{B(y, \widetilde{r}/\widetilde{A})} \leq M_\rho(f)(y)$ together with (1), we have that

$$\left[f_B\chi_{3B} + f\chi_{(3B)^\complement}\right]_{\widetilde{B}} - M_\rho(f)(y)$$
$$\leq \left[\left(f_B - f_{3\widetilde{B}}\right)\chi_{3B} + \left(f - f_{3\widetilde{B}}\right)\chi_{(3B)^\complement}\right]_{\widetilde{B}} + \left|f_{3\widetilde{B}} - f_{B(y, \widetilde{r}/\widetilde{A})}\right| \lesssim 1.$$

Combining the two inequalities above and (16) leads to that

$$[M_\rho(f_B\chi_{3B} + f\chi_{(3B)^\complement})]_B - \operatorname{essinf}_B M_\rho(f) \lesssim 1,$$

which together with (15) further implies (14). This finishes the proof of Theorem 4.1. $\square$

Lemma 4.1. *Let ρ be an admissible function on $\mathcal{X}$. Then $f \in \mathrm{BLO}_\rho(\mathcal{X})$ if and only if $f \in L^1_{\mathrm{loc}}(\mathcal{X})$, $M_\rho(f) - f \in L^\infty(\mathcal{X})$ and*

$$\sup_{B \in \mathcal{D}} \frac{1}{\mu(B)} \int_B |f(y)| \, d\mu(y) \lesssim 1. \tag{17}$$

Furthermore, $\|M_\rho(f) - f\|_{L^\infty(\mathcal{X})} \sim \|f\|_{\mathrm{BLO}_\rho(\mathcal{X})}$.

Proof. Assuming that $f \in \mathrm{BLO}_\rho(\mathcal{X})$, we then see that (17) holds. Since μ is regular, for μ-a. e. $x \in \mathcal{X}$, there exists a sequence of balls $\{B_k\}_k$ centered at x with $r_{B_k} \to 0$ as $k \to \infty$ such that

$$\lim_{k \to \infty} \frac{1}{\mu(B_k)} \int_{B_k} f(y) \, d\mu(y) = f(x). \tag{18}$$

Let x be any point satisfying (18) and B be a ball containing x with $B \notin \mathcal{D}$. Then we obtain that $f(x) \geq \operatorname{essinf}_B f$ and $f_B - f(x) \leq \|f\|_{\mathrm{BLO}_\rho(\mathcal{X})}$. Taking the supremum over all balls B containing x and $B \notin \mathcal{D}$, we have $M_\rho(f)(x) - f(x) \leq \|f\|_{\mathrm{BLO}_\rho(\mathcal{X})}$.

Conversely, assume that f satisfies (17) and $M_\rho(f) - f \in L^\infty(\mathcal{X})$. Then for all balls $B \notin \mathcal{D}$ and μ-a. e. $x \in B$, $f(x) \geq f_B - \|M_\rho(f) - f\|_{L^\infty(\mathcal{X})}$. This yields that

$$\operatorname*{essinf}_B f \geq f_B - \|M_\rho(f) - f\|_{L^\infty(\mathcal{X})},$$

which together with (17) implies that $f \in \mathrm{BLO}_\rho(\mathcal{X})$ and $\|f\|_{\mathrm{BLO}_\rho(\mathcal{X})} \lesssim \|M_\rho(f) - f\|_{L^\infty(\mathcal{X})}$. This finishes the proof of Lemma 4.1. $\qquad\square$

Theorem 4.2. *Let ρ be an admissible function on $\mathcal{X}$. Then $f \in \mathrm{BLO}_\rho(\mathcal{X})$ if and only if there exist $h \in L^\infty(\mathcal{X})$ and $g \in \mathrm{BMO}_\rho(\mathcal{X})$ such that*

$$f = M_\rho(g) + h. \tag{19}$$

Furthermore, $\|f\|_{\mathrm{BLO}_\rho(\mathcal{X})} \sim \inf(\|g\|_{\mathrm{BMO}_\rho(\mathcal{X})} + \|h\|_{L^\infty(\mathcal{X})})$, where the infimum is taken over all representations of f as in (19).

Proof. If there exist g and h satisfying (19), then by Theorem 4.1, $M_\rho(g) \in \mathrm{BLO}_\rho(\mathcal{X})$, which together with $L^\infty(\mathcal{X}) \subset \mathrm{BLO}_\rho(\mathcal{X})$ implies that $f \in \mathrm{BLO}_\rho(\mathcal{X})$ and

$$\|f\|_{\mathrm{BLO}_\rho(\mathcal{X})} \lesssim \|M_\rho(g)\|_{\mathrm{BLO}_\rho(\mathcal{X})} + \|h\|_{L^\infty(\mathcal{X})} \lesssim \|g\|_{\mathrm{BMO}_\rho(\mathcal{X})} + \|h\|_{L^\infty(\mathcal{X})}.$$

To see the converse, assume that $f \in \mathrm{BLO}_\rho(\mathcal{X})$. By $\mathrm{BLO}_\rho(\mathcal{X}) \subset \mathrm{BMO}_\rho(\mathcal{X})$ and Theorem 4.1, we see $M_\rho(f) \in \mathrm{BLO}_\rho(\mathcal{X})$. Let $h \equiv f - M_\rho(f)$ and $g \equiv f$. Then Theorem 4.2 follows from Lemma 4.1, which completes the proof of Theorem 4.2. $\qquad\square$

As another corollary of Theorem 4.1, we obtain the boundedness of HL, HL_ρ and M from $\mathrm{BMO}_\rho(\mathcal{X})$ to $\mathrm{BLO}_\rho(\mathcal{X})$. To this end, we first establish the following useful lemma.

Lemma 4.2. *Let ρ be an admissible function on $\mathcal{X}$ and Y be one of the spaces $\mathrm{BMO}(\mathcal{X})$, $\mathrm{BMO}_\rho(\mathcal{X})$, $\mathrm{BLO}(\mathcal{X})$ and $\mathrm{BLO}_\rho(\mathcal{X})$. If $f \in Y$, $h_1, h_2 \in L^\infty(\mathcal{X})$, and $f - h_2 \leq g \leq f + h_1$, then $g \in Y$ and*

$$\|g\|_Y \leq \|f\|_Y + \|h_1\|_{L^\infty(\mathcal{X})} + \|h_2\|_{L^\infty(\mathcal{X})}.$$

Proof. We only consider the case that $Y = \mathrm{BLO}_\rho(\mathcal{X})$ by similarity. For all balls $B \in \mathcal{D}$, we have that

$$|g|_B \leq |f|_B + \|h_1\|_{L^\infty(\mathcal{X})} + \|h_2\|_{L^\infty(\mathcal{X})} \leq \|f\|_{\mathrm{BLO}_\rho} + \|h_1\|_{L^\infty(\mathcal{X})} + \|h_2\|_{L^\infty(\mathcal{X})}.$$

On the other hand, for all balls $B \notin \mathcal{D}$,

$$g_B - \operatornamewithlimits{essinf}_B g \leq f_B + \|h_1\|_{L^\infty(\mathcal{X})} - \operatornamewithlimits{essinf}_B f + \|h_2\|_{L^\infty(\mathcal{X})} \leq \|f\|_{\mathrm{BLO}_\rho} + \|h_1\|_{L^\infty(\mathcal{X})} + \|h_2\|_{L^\infty(\mathcal{X})}.$$

Combining the two inequalities above finishes the proof of Lemma 4.2. $\square$

Corollary 4.1. *Let ρ be an admissible function on $\mathcal{X}$. Then* HL*,* HL_ρ *and* M *are bounded from* $\mathrm{BMO}_\rho(\mathcal{X})$ *to* $\mathrm{BLO}_\rho(\mathcal{X})$*, namely, there exists a positive constant C such that for all $f \in \mathrm{BMO}_\rho(\mathcal{X})$,* $\mathrm{HL}(f)$*,* $\mathrm{HL}_\rho(f)$*,* $M(f) \in \mathrm{BLO}_\rho(\mathcal{X})$ *and*

$$\|\mathrm{HL}(f)\|_{\mathrm{BLO}_\rho(\mathcal{X})} + \|\mathrm{HL}_\rho(f)\|_{\mathrm{BLO}_\rho(\mathcal{X})} + \|M(f)\|_{\mathrm{BLO}_\rho(\mathcal{X})} \leq C\|f\|_{\mathrm{BMO}_\rho(\mathcal{X})}.$$

Proof. Since for all locally integrable functions f,

$$M_\rho(f) \leq M(f) \leq M_\rho(f) + \|f\|_{\mathrm{BMO}_\rho(\mathcal{X})},$$

by Theorem 4.1 and Lemma 4.2, we have that if $f \in \mathrm{BMO}_\rho(\mathcal{X})$, then $M(f) \in \mathrm{BLO}_\rho(\mathcal{X})$ and $\|M(f)\|_{\mathrm{BLO}_\rho(\mathcal{X})} \lesssim \|f\|_{\mathrm{BMO}_\rho(\mathcal{X})}$. Using this together with Theorem 4.1, the facts that $\mathrm{HL}_\rho(f) = M_\rho(|f|)$, $\mathrm{HL}(f) = M(|f|)$ and that $\||f|\|_{\mathrm{BMO}_\rho(\mathcal{X})} \leq 2\|f\|_{\mathrm{BMO}_\rho(\mathcal{X})}$ for all $f \in \mathrm{BMO}_\rho(\mathcal{X})$, we have that $\mathrm{HL}(f)$, $\mathrm{HL}_\rho(f) \in \mathrm{BLO}_\rho(\mathcal{X})$ and

$$\|\mathrm{HL}_\rho(f)\|_{\mathrm{BLO}_\rho(\mathcal{X})} + \|\mathrm{HL}(f)\|_{\mathrm{BLO}_\rho(\mathcal{X})} \lesssim \|f\|_{\mathrm{BMO}_\rho(\mathcal{X})}.$$

This finishes the proof of Corollary 4.1. $\square$

Remark 4.1. Let $d \geq 3$, V be a nonnegative integrable function on $\mathbb{R}^d$ and $\mathcal{L} = -\Delta + V$. If $q > d/2$, $V \in \mathcal{B}_q(\mathbb{R}^d)$ and ρ is as in (4), then $\mathrm{BMO}_\rho(\mathbb{R}^d)$ is just the space $\mathrm{BMO}_\mathcal{L}(\mathbb{R}^d)$ introduced in [9]. It was proved in [9] that HL is bounded on $\mathrm{BMO}_\rho(\mathbb{R}^d)$. Recall that $\mathrm{BLO}_\rho(\mathbb{R}^d) \subsetneq \mathrm{BMO}_\rho(\mathbb{R}^d)$. Thus, Corollary 4.1 improves the result of [9]. Similar claim is also true for HL on Heisenberg groups; see [22].

5. **Boundedness of several maximal operators.** This section consists of two subsections. Subsection 5.1 is devoted to the boundedness of several radial maximal operators from $\mathrm{BMO}_\rho(\mathcal{X})$ to $\mathrm{BLO}_\rho(\mathcal{X})$; while in Subsection 5.2, we obtain the boundedness of the Poisson semigroup maximal operator from $\mathrm{BMO}_\rho(\mathcal{X})$ to $\mathrm{BLO}_\rho(\mathcal{X})$.

5.1. **Boundedness of radial maximal operators.**

Theorem 5.1. *Let ρ be an admissible function on $\mathcal{X}$ and $S^+(f)$ be as in Definition 2.4. Then there exists a positive constant C such that for all $f \in \mathrm{BMO}_\rho(\mathcal{X})$,*

$$\|S^+(f)\|_{\mathrm{BLO}_\rho(\mathcal{X})} \leq C\|f\|_{\mathrm{BMO}_\rho(\mathcal{X})}.$$

Proof. By the homogeneity of $\|\cdot\|_{\mathrm{BMO}_\rho(\mathcal{X})}$ and $\|\cdot\|_{\mathrm{BLO}_\rho(\mathcal{X})}$, we may assume that

$$\|f\|_{\mathrm{BMO}_\rho(\mathcal{X})} = 1.$$

Observe that by Definition 2.3 (i), $S^+(f) \lesssim \mathrm{HL}(f)$. From this and Corollary 4.1, it follows that for all balls $B \equiv B(x_0,\, r) \in \mathcal{D}$,

$$\frac{1}{\mu(B)} \int_B S^+(f)(x)\, d\mu(x) \lesssim 1.$$

This also implies that $S^+(f)(x) < \infty$ for μ-a. e. $x \in \mathcal{X}$. Moreover, by the inequality above, to finish the proof Theorem 5.1, it suffices to show that for all balls $B \equiv B(x_0,\, r) \notin \mathcal{D}$ and $y \in B$,

$$\frac{1}{\mu(B)} \int_B [S^+(f)(x) - S^+(f)(y)]\, d\mu(x) \lesssim 1. \tag{20}$$

Let $f_1 \equiv (f - f_B)\chi_{2B}$, $f_2 \equiv (f - f_B)\chi_{(2B)^\complement}$, $B_1 \equiv \{x \in B :\ S_r^+(f)(x) \geq S_\infty^+(f)(x)\}$ and $B_2 \equiv B \setminus B_1$, where for all $x \in \mathcal{X}$, $S_r^+(f)(x) \equiv \sup_{0 < t < r} |S_t(f)(x)|$ and $S_\infty^+(f)(x) \equiv \sup_{r \leq t < \infty} |S_t(f)(x)|$. By using Definition 2.3 (iv), we have that for all $y \in B$,

$$\frac{1}{\mu(B)} \int_B [S^+(f)(x) - S^+(f)(y)]\, d\mu(x)$$
$$= \frac{1}{\mu(B)} \int_{B_1} \left[S_r^+(f)(x) - S^+(f)(y) \right]\, d\mu(x)$$
$$+ \frac{1}{\mu(B)} \int_{B_2} \left[S_\infty^+(f)(x) - S^+(f)(y) \right]\, d\mu(x)$$
$$\leq \frac{1}{\mu(B)} \int_B S_r^+(f_1)(x)\, d\mu(x) + \frac{1}{\mu(B)} \int_B S_r^+(f_2)(x)\, d\mu(x)$$
$$+ |f_B - S_r(f)(y)| + \frac{1}{\mu(B)} \int_{B_2} \left[S_\infty^+(f)(x) - S_\infty^+(f)(y) \right]\, d\mu(x)$$
$$\equiv \mathrm{L}_1 + \mathrm{L}_2 + \mathrm{L}_3 + \mathrm{L}_4.$$

By the Hölder inequality, $S^+(f) \lesssim \mathrm{HL}(f)$, the $L^2(\mathcal{X})$-boundedness of HL, (1) and Lemma 3.1, we obtain

$$\mathrm{L}_1 \lesssim \left(\frac{1}{\mu(B)} \int_B [\mathrm{HL}(f_1)(x)]^2\, d\mu(x) \right)^{1/2} \lesssim \left(\frac{1}{\mu(B)} \int_{2B} |f(x) - f_{2B}|^2\, d\mu(x) \right)^{1/2} \lesssim 1.$$

Recall that $\{S_t\}_{t>0}$ is a continuous (ϵ_1, ϵ_2)-AOTI. By Remark 2.1, Definition 2.3 (i), (1) and the fact that for all $x \in B$ and $j \in \mathbb{N}$, $2^{j+1}B \subset B(x, 2^{j+2}r)$, we have that for all $t \in (0, r)$,

$$|S_t(f_2)(x)| \lesssim \int_{(2B)^\complement} \frac{1}{V(x, z)} \left(\frac{t}{t + d(x, z)} \right)^{\epsilon_2} |f(z) - f_B|\, d\mu(z)$$
$$\lesssim \sum_{j=1}^\infty \left(\frac{t}{2^{j-1}r} \right)^{\epsilon_2} \frac{1}{V_{2^{j-1}r}(x)} \int_{2^{j+1}B} |f(z) - f_B|\, d\mu(z)$$

$$\lesssim \left(\frac{t}{r}\right)^{\epsilon_2} \sum_{j=1}^{\infty} 2^{-j\epsilon_2} \left[\frac{1}{\mu(2^{j+1}B)} \int_{2^{j+1}B} |f(z) - f_{2^{j+1}B}| \, d\mu(z) + |f_B - f_{2^{j+1}B}|\right]$$

$$\lesssim \left(\frac{t}{r}\right)^{\epsilon_2} \sum_{j=1}^{\infty} 2^{-j\epsilon_2}(j+1) \lesssim 1,$$

which implies that $\mathrm{L}_2 \lesssim 1$.

By Definition 2.3 (iv) and (i) together with (1) and the fact that for all $y \in B$ and $j \in \mathbb{N} \cup \{0\}$, $2^{j+1}B \subset B(y, 2^{j+2}r)$, we have

$$\mathrm{L}_3 \leq \int_{\mathcal{X}} |S_r(y, z)[f(z) - f_B]| \, d\mu(z)$$

$$\lesssim \int_{\mathcal{X}} \frac{1}{V_r(y) + V(y, z)} \left(\frac{r}{r + d(y, z)}\right)^{\epsilon_2} |f(z) - f_B| \, d\mu(z)$$

$$\lesssim \sum_{j=0}^{\infty} 2^{-j\epsilon_2} \frac{1}{V_{2^{j-1}r}(y)} \int_{2^{j+1}B} |f(z) - f_B| \, d\mu(z) \lesssim \sum_{j=0}^{\infty} 2^{-j\epsilon_2}(1+j) \lesssim 1.$$

On the other hand, for all $x, y \in B$ and $t \in [r, \infty)$, $B(x, t) \subset B(y, 2t) \subset B(x, 3t)$. It follows from this fact and (1) that

$$\left|f_{B(x,t)} - f_{B(y,t)}\right| \lesssim \frac{1}{\mu(B(x, 3t))} \int_{B(y, 2t)} \left|f(z) - f_{B(y,t)}\right| \, d\mu(z) \lesssim 1.$$

By this and an argument similar to the estimate for L_3, we have that for all $x, y \in B$ and $t \in [r, \infty)$,

$$|S_t(f)(x) - S_t(f)(y)|$$
$$\lesssim \left|S_t(f)(x) - f_{B(x,t)}\right| + \left|f_{B(x,t)} - f_{B(y,t)}\right| + \left|f_{B(y,t)} - S_t(f)(y)\right| \lesssim 1,$$

which implies that

$$\mathrm{L}_4 \lesssim \frac{1}{\mu(B)} \int_B \sup_{r \leq t < \infty} |S_t(f)(x) - S_t(f)(y)| \, d\mu(x) \lesssim 1.$$

Combining the estimates for L_1 through L_4 yields (20), which completes the proof of Theorem 5.1. $\qquad\square$

By Definition 2.3 (i), we have that for all $x \in \mathcal{X}$ and $t \in [\rho(x), \infty)$,

$$|S_t(f)(x)| \lesssim \int_{\mathcal{X}} \frac{1}{V_t(x) + V(x, y)} \left(\frac{t}{t + d(x, y)}\right)^{\epsilon_2} |f(y)| \, d\mu(y)$$

$$\lesssim \sum_{j=0}^{\infty} 2^{-j\epsilon_2} \frac{1}{V_{2^{j-1}t}(x)} \int_{d(x,y)<2^j t} |f(y)| \, d\mu(y) \lesssim \|f\|_{\mathrm{BMO}_\rho(\mathcal{X})}. \qquad (21)$$

This implies that there exists a positive constant $\widetilde{C}$ such that for all $f \in \mathrm{BMO}_\rho(\mathcal{X})$,

$$S_\rho^+(f) \leq S^+(f) \leq S_\rho^+(f) + \widetilde{C}\|f\|_{\mathrm{BMO}_\rho(\mathcal{X})}.$$

From this, Lemma 4.2 and Theorem 5.1, we deduce the following corollary.

Corollary 5.1. *Let ρ be an admissible function on $\mathcal{X}$ and $S_\rho^+(f)$ be as in Definition 2.4. Then there exists a positive constant C such that for all $f \in \mathrm{BMO}_\rho(\mathcal{X})$, $S_\rho^+(f) \in \mathrm{BLO}_\rho(\mathcal{X})$ and $\|S_\rho^+(f)\|_{\mathrm{BLO}_\rho(\mathcal{X})} \le C\|f\|_{\mathrm{BMO}_\rho(\mathcal{X})}$.*

Let $\{T_t\}_{t>0}$ be a family of bounded linear operators with integral kernels $\{T_t(x, y)\}_{t>0}$. Assume that there exist constants $C \in (0, \infty)$, $\epsilon_1 \in (0, 1]$, $\epsilon_2 \in (0, \infty)$, $\delta \in (0, 1]$ and $\gamma \in (0, \infty)$, and an (ϵ_1, ϵ_2)-AOTI $\{\widetilde{T}_t\}_{t>0}$ with kernels $\{\widetilde{T}_t(x,y)\}_{t>0}$ such that for all $t \in (0, \infty)$ and $x, y \in \mathcal{X}$,

$$\left| T_t(x, y) - \widetilde{T}_t(x, y) \right| \le C \left(\frac{t}{t + \rho(x)} \right)^\delta \frac{1}{V_t(x) + V(x, y)} \left(\frac{t}{t + d(x, y)} \right)^\gamma. \tag{22}$$

Notice that by (22), for all $f \in \mathrm{BMO}_\rho(\mathcal{X})$ and $t \in (0, \infty)$, we have that for all $x \in \mathcal{X}$,

$$\begin{aligned}
\left| T_t(f)(x) - \widetilde{T}_t(f)(x) \right| &\lesssim \left(\frac{t}{t + \rho(x)} \right)^\delta \sum_{j=1}^\infty 2^{-j\gamma} \frac{1}{V_{2^{j-1}t}(x)} \int_{d(x, y) < 2^j t} f(y) \, d\mu(y) \\
&\lesssim \|f\|_{\mathrm{BMO}_\rho(\mathcal{X})} \left(\frac{t}{t + \rho(x)} \right)^\delta \sum_{j=1}^\infty 2^{-j\gamma} \log \left(1 + \frac{\rho(x)}{2^j t} \right) \\
&\lesssim \|f\|_{\mathrm{BMO}_\rho(\mathcal{X})}. \tag{23}
\end{aligned}$$

Define the maximal operators T^+ and $\widetilde{T}^+$ as in Definition 2.4 (i) with S_t replaced by T_t and $\widetilde{T}_t$, respectively. Then by (23), there exists a positive constant C such that for all $f \in \mathrm{BMO}_\rho(\mathcal{X})$,

$$\widetilde{T}^+(f) - C\|f\|_{\mathrm{BMO}_\rho(\mathcal{X})} \le T^+(f) \le \widetilde{T}^+(f) + C\|f\|_{\mathrm{BMO}_\rho(\mathcal{X})}.$$

Since $\widetilde{T}^+$ is bounded from $\mathrm{BMO}_\rho(\mathcal{X})$ to $\mathrm{BLO}_\rho(\mathcal{X})$ (see Theorem 5.1), applying Lemma 4.2 again, we have the following corollary.

Corollary 5.2. *Assume that (22) holds. Then there exists a positive constant C such that for all $f \in \mathrm{BMO}_\rho(\mathcal{X})$, $T^+(f) \in \mathrm{BMO}_\rho(\mathcal{X})$ and $\|T^+(f)\|_{\mathrm{BLO}_\rho(\mathcal{X})} \le C\|f\|_{\mathrm{BMO}_\rho(\mathcal{X})}$.*

5.2. **Boundedness of Poisson semigroup maximal functions.** Let $\{T_t\}_{t>0}$ be a family of bounded linear integral operators on $L^2(\mathcal{X})$ and

$$P_t(f) \equiv \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-s}}{\sqrt{s}} T_{t/(2\sqrt{s})}(f) \, ds. \tag{24}$$

Define the maximal operator P^+ as in Definition 2.4 (i) by replacing T_t with P_t. If $\{T_t\}_{t>0}$ is replaced by another family $\{\widetilde{T}_t\}_{t>0}$ of bounded linear integral operators on $L^2(\mathcal{X})$, we then denote the corresponding P_t in (24) by $\widetilde{P}_t$ and the corresponding maximal operator by $\widetilde{P}^+$.

Lemma 5.1. *(i) Let $\{\widetilde{T}_t\}_{t>0}$ be an (ϵ_1, ϵ_2)-AOTI for $\epsilon_1 \in (0, 1]$ and $\epsilon_2 \in (0, \infty)$. Then $\{\widetilde{P}_t\}_{t>0}$ is an $(\epsilon_1, \epsilon_2')$-AOTI with $\epsilon_2' \in (0, \epsilon_2] \cap (0, 1)$.*

(ii) If $\{\widetilde{T}_t\}_{t>0}$ and $\{T_t\}_{t>0}$ satisfy (22) with constants $C \in (0, \infty)$, $\epsilon_1 \in (0, 1]$, $\epsilon_2 \in (0, \infty)$, $\delta \in (0, 1]$ and $\gamma \in (0, \infty)$, then so do $\{\widetilde{P}_t\}_{t>0}$ and $\{P_t\}_{t>0}$ with constants $C' \in (0, \infty)$, $\epsilon_1, \epsilon_2', \delta'$ and γ', where $\epsilon_2' \in (0, \epsilon_2] \cap (0, 1)$, $\gamma' \in (0, \gamma)$ and $\delta' \in (0, \delta)$ satisfying $0 < \gamma' + \delta' < 1$.

Proof. We first prove (i). By $\int_{\mathcal{X}} \widetilde{T}_t(x, y) \, d\mu(y) = 1$ for all $x \in \mathcal{X}$, we obtain

$$\int_{\mathcal{X}} \widetilde{P}_t(x, y) \, d\mu(y) = \frac{1}{\sqrt{\pi}} \int_{\mathcal{X}} \int_0^\infty \frac{e^{-s}}{\sqrt{s}} \widetilde{T}_{t/(2\sqrt{s})}(x, y) \, ds \, d\mu(y) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-s}}{\sqrt{s}} \, ds = 1.$$

Similarly, for any $x \in \mathcal{X}$, it follows from $\int_{\mathcal{X}} \widetilde{T}_t(y, x) \, d\mu(y) = 1$ that $\int_{\mathcal{X}} \widetilde{P}_t(y, x) \, d\mu(y) = 1$.

For all $s, t \in (0, \infty)$ and $x, y \in \mathcal{X}$, from the fact that

$$t + d(x, y) \leq (1 + s)[t/s + d(x, y)], \tag{25}$$

it follows that

$$\frac{t/s}{t/s + d(x, y)} \leq (1 + s^{-1}) \frac{t}{t + d(x, y)}. \tag{26}$$

On the other hand, by (1) and (25), we have that for all $s, t \in (0, \infty)$ and $x, y \in \mathcal{X}$,

$$V_{t/s}(x) + V(x, y) \sim \mu(B(x, t/s + d(x, y)))$$
$$\gtrsim (1 + s)^{-n} \mu(B(x, t + d(x, y))) \sim (1 + s)^{-n} [V_t(x) + V(x, y)]. \tag{27}$$

Since $\{\widetilde{T}_t\}_{t>0}$ is an (ϵ_1, ϵ_2)-AOTI, by Definition 2.3 (i), (26) and (27), we obtain that for all $\epsilon_2' \in (0, \epsilon_2]$ and all $x, y \in \mathcal{X}$,

$$\left| \widetilde{P}_t(x, y) \right| \lesssim \int_0^\infty e^{-s^2/4} \left| \widetilde{T}_{t/s}(x, y) \right| ds$$
$$\lesssim \int_0^\infty e^{-s^2/4} \frac{1}{V_{t/s}(x) + V(x, y)} \left(\frac{t/s}{t/s + d(x, y)} \right)^{\epsilon_2} ds$$
$$\lesssim \frac{1}{V_t(x) + V(x, y)} \left(\frac{t}{t + d(x, y)} \right)^{\epsilon_2'}. \tag{28}$$

Now we prove that for all $x, x', y \in \mathcal{X}$ and $t \in (0, \infty)$, if $d(x, x') \leq \frac{1}{2}[t + d(x, y)]$, then

$$\left| \widetilde{P}_t(x, y) - \widetilde{P}_t(x', y) \right| \lesssim \left(\frac{d(x, x')}{t + d(x, y)} \right)^{\epsilon_1} \frac{1}{V_t(x) + V(x, y)} \left(\frac{t}{t + d(x, y)} \right)^{\epsilon_2'}. \tag{29}$$

If $\frac{1}{8}[t + d(x, y)] \leq d(x, x') \leq \frac{1}{2}[t + d(x, y)]$, (28) yields (29). If $\frac{1}{4} d(x, y) \leq d(x, x') < \frac{1}{8}[t + d(x, y)]$, then $d(x, y) \leq 4 d(x, x') \leq t$. In this case, $t + d(x, y) \sim t + d(x', y)$. By Definition 2.3 (ii), (25), (26) and (27), we obtain

$$\left| \widetilde{P}_t(x, y) - \widetilde{P}_t(x', y) \right|$$

$$\lesssim \int_0^\infty e^{-s^2/4} \left| \widetilde{T}_{t/s}(x,\, y) - \widetilde{T}_{t/s}(x',\, y) \right| \, ds$$

$$\lesssim \left[\int_0^{t/[2d(x,x')]} \left(\frac{d(x,\, x')}{t/s + d(x,\, y)} \right)^{\epsilon_1} + \int_{t/[2d(x,x')]}^\infty \right] e^{-s^2/4} \frac{1}{V_{t/s}(x) + V(x,\, y)}$$

$$\times \left(\frac{t/s}{t/s + d(x,\, y)} \right)^{\epsilon_2} \, ds$$

$$\lesssim \left(\frac{d(x,\, x')}{t + d(x,\, y)} \right)^{\epsilon_1} \frac{1}{V_t(x) + V(x,\, y)} \left(\frac{t}{t + d(x,\, y)} \right)^{\epsilon_2'} \left[\int_0^{t/[2d(x,x')]} (1+s)^{\epsilon_1} \right.$$

$$\left. + \int_{t/[2d(x,x')]}^\infty s^{\epsilon_1} \right] e^{-s^2/4} (1+s)^n (1 + s^{-\epsilon_2'}) \, ds$$

$$\lesssim \left(\frac{d(x,\, x')}{t + d(x,\, y)} \right)^{\epsilon_1} \frac{1}{V_t(x) + V(x,\, y)} \left(\frac{t}{t + d(x,\, y)} \right)^{\epsilon_2'}.$$

If $d(x,\, x') < \frac{1}{4} d(x,\, y)$, then by Definition 2.3 (ii), (25), (26) and (27), we have

$$|\widetilde{P}_t(x,\, y) - \widetilde{P}_t(x',\, y)|$$

$$\lesssim \int_0^\infty e^{-s^2/4} \left| \widetilde{T}_{t/s}(x,\, y) - \widetilde{T}_{t/s}(x',\, y) \right| \, ds$$

$$\lesssim \int_0^\infty e^{-s^2/4} \left(\frac{d(x,\, x')}{t/s + d(x,\, y)} \right)^{\epsilon_1} \frac{1}{V_{t/s}(x) + V(x,\, y)} \left(\frac{t/s}{t/s + d(x,\, y)} \right)^{\epsilon_2} \, ds$$

$$\lesssim \left(\frac{d(x,\, x')}{t + d(x,\, y)} \right)^{\epsilon_1} \frac{1}{V_t(x) + V(x,\, y)} \left(\frac{t}{t + d(x,\, y)} \right)^{\epsilon_2'},$$

which verifies (29). Similarly with x and y interchanged, we have that $\widetilde{P}_t(x,\, y)$ satisfies Definition 2.3 (iii). This shows (i).

To prove (ii), by (i), we only need to prove (22) for $\widetilde{P}_t$ and P_t. By (22), (25), (26) with $d(x,\, y)$ replaced by $\rho(x)$, and (27), we have that for all $\gamma' \in (0,\, \gamma)$ and $\delta' \in (0,\, \delta')$ satisfying $0 < \gamma' + \delta' < 1$ and $x,\, y \in \mathcal{X}$,

$$|\widetilde{P}_t(x,\, y) - P_t(x,\, y)|$$

$$\lesssim \int_0^\infty \frac{e^{-s}}{\sqrt{s}} \left| \widetilde{T}_{t/(2\sqrt{s})}(x,\, y) - T_{t/(2\sqrt{s})}(x,\, y) \right| \, ds$$

$$\lesssim \int_0^\infty e^{-4s^2} \left(\frac{t/s}{t/s + \rho(x)} \right)^\delta \frac{1}{V_{t/s}(x) + V(x,\, y)} \left(\frac{t/s}{t/s + d(x,\, y)} \right)^\gamma \, ds$$

$$\lesssim \left(\frac{t}{t + \rho(x)} \right)^{\delta'} \frac{1}{V_t(x) + V(x,\, y)} \left(\frac{t}{t + d(x,\, y)} \right)^{\gamma'}.$$

This finishes the proof of Lemma 5.1. $\square$

Then by Theorem 5.1, Corollaries 5.1 and 5.2, we obtain the following result.

Theorem 5.2. *Let ρ be an admissible function. Assume that $\{\widetilde{T}_t\}_{t>0}$ and $\{T_t\}_{t>0}$ satisfy (22) with constants $C \in (0, \infty)$, $\epsilon_1 \in (0, 1]$, $\epsilon_2 \in (0, \infty)$, $\delta \in (0, 1]$ and $\gamma \in (0, \infty)$. Then there exists a positive constant C such that for all $f \in \mathrm{BMO}_\rho(\mathcal{X})$, $P^+(f)$, $\widetilde{P}^+(f)$, $\widetilde{P}_\rho^+(f) \in \mathrm{BLO}_\rho(\mathcal{X})$ and*

$$\|P^+(f)\|_{\mathrm{BLO}_\rho(\mathcal{X})} + \|\widetilde{P}^+(f)\|_{\mathrm{BLO}_\rho(\mathcal{X})} + \|\widetilde{P}_\rho^+(f)\|_{\mathrm{BLO}_\rho(\mathcal{X})} \leq C\|f\|_{\mathrm{BMO}_\rho(\mathcal{X})}.$$

6. **Boundedness of the Littlewood-Paley g-function.** In this section, we consider the boundedness of certain variant of the Littlewood-Paley g-function on $\mathrm{BMO}_\rho(\mathcal{X})$.

Let ρ be an admissible function on $\mathcal{X}$ and $\{Q_t\}_{t>0}$ be a family of operators bounded on $L^2(\mathcal{X})$ with integral kernels $\{Q_t(x, y)\}_{t>0}$ satisfying that there exist constants $C \in (0, \infty)$, $\delta_1 \in (0, \infty)$, $\beta \in (0, 1]$, $\delta_2 \in (0, 1)$ and $\gamma \in (0, \infty)$ such that for all $t \in (0, \infty)$ and $x, x', y \in \mathcal{X}$ with $d(x, x') \leq \frac{t}{2}$,

$(Q)_i$ $|Q_t(x, y)| \leq C \frac{1}{V_t(x) + V(x, y)} \left(\frac{t}{t + d(x, y)}\right)^\gamma \left(\frac{\rho(x)}{t + \rho(x)}\right)^{\delta_1}$;

$(Q)_{ii}$ $|Q_t(x, y) - Q_t(x', y)| \leq C \left(\frac{d(x, x')}{t + d(x, y)}\right)^\beta \frac{1}{V_t(x) + V(x, y)} \left(\frac{t}{t + d(x, y)}\right)^\gamma$;

$(Q)_{iii}$ $|\int_{\mathcal{X}} Q_t(x, y) d\mu(y)| \leq C \left(\frac{t}{t + \rho(x)}\right)^{\delta_2}$.

For all $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ and $x \in \mathcal{X}$, define the Littlewood-Paley g-function by

$$g(f)(x) \equiv \left(\int_0^\infty |Q_t(f)(x)|^2 \frac{dt}{t}\right)^{1/2}. \tag{30}$$

Theorem 6.1. *Let ρ be an admissible function on $\mathcal{X}$. Suppose the g-function defined in (30) is bounded on $L^2(\mathcal{X})$. Then there exists a positive constant C such that for all $f \in \mathrm{BMO}_\rho(\mathcal{X})$, $[g(f)]^2 \in \mathrm{BLO}_\rho(\mathcal{X})$ and $\|[g(f)]^2\|_{\mathrm{BLO}_\rho(\mathcal{X})} \leq C\|f\|^2_{\mathrm{BMO}_\rho(\mathcal{X})}$.*

Proof. By the homogeneity of $\|\cdot\|_{\mathrm{BMO}_\rho(\mathcal{X})}$ and $\|\cdot\|_{\mathrm{BLO}_\rho(\mathcal{X})}$, we assume that $f \in \mathrm{BMO}_\rho(\mathcal{X})$ and $\|f\|_{\mathrm{BMO}_\rho(\mathcal{X})} = 1$. We first prove that for all balls $B \equiv B(x_0, r)$ with $r \geq \rho(x_0)$,

$$\frac{1}{\mu(B)} \int_B [g(f)(x)]^2 \, d\mu(x) \lesssim 1. \tag{31}$$

For any $x \in B$, write

$$[g(f)(x)]^2 \equiv \int_0^{8\rho(x)} |Q_t(f)(x)|^2 \frac{dt}{t} + \int_{8\rho(x)}^\infty |Q_t(f)(x)|^2 \frac{dt}{t} \equiv [g_1(f)(x)]^2 + [g_2(f)(x)]^2.$$

By the $L^2(\mathcal{X})$-boundedness of g, (1) and Lemma 3.1, we have

$$\frac{1}{\mu(B)} \int_B [g_1(f\chi_{2B})(x)]^2 \, d\mu(x) \lesssim \frac{1}{\mu(B)} \int_{2B} |f(x)|^2 \, d\mu(x) \lesssim 1. \tag{32}$$

For any $x \in B$, by $(Q)_i$,

$$\left|Q_t\left(f\chi_{(2B)^\complement}\right)(x)\right| \lesssim \int_{(2B)^\complement} \frac{1}{V_t(x) + V(x, y)} \left(\frac{t}{t + d(x, y)}\right)^\gamma |f(y)| \, d\mu(y)$$

$$\lesssim \left(\frac{t}{r}\right)^\gamma \sum_{j=1}^\infty \frac{2^{-j\gamma}}{V_{2^{j-1}r}(x)} \int_{d(x,\,y)<2^j r} |f(y)|\,d\mu(y) \lesssim \left(\frac{t}{r}\right)^\gamma.$$

Notice that for all $x \in B$, by (10), we have $\rho(x) \lesssim r$. From the inequality above we deduce that

$$\frac{1}{\mu(B)} \int_B \left[g_1\left(f\chi_{(2B)^\complement}\right)(x)\right]^2 d\mu(x) \lesssim \frac{1}{\mu(B)} \int_B \int_0^{8\rho(x)} \left(\frac{t}{r}\right)^{2\gamma} \frac{dt}{t}\,d\mu(x) \lesssim 1. \qquad (33)$$

Combining (32) and (33) gives us that

$$\frac{1}{\mu(B)} \int_B [g_1(f)(x)]^2\,d\mu(x) \lesssim 1. \qquad (34)$$

To prove (31) with g_2, we first notice that for all $x \in B$ and $t \geq 8\rho(x)$,

$$|Q_t(f)(x)| \lesssim \int_\mathcal{X} \frac{1}{V_t(x) + V(x,\,y)} \left(\frac{t}{t + d(x,\,y)}\right)^\gamma \left(\frac{\rho(x)}{t + \rho(x)}\right)^{\delta_1} |f(y)|\,d\mu(y)$$

$$\lesssim \left(\frac{\rho(x)}{t}\right)^{\delta_1} \sum_{j=0}^\infty 2^{-j\gamma} \frac{1}{V_{2^{j-1}t}(x)} \int_{d(x,\,y)<2^j t} |f(y)|\,d\mu(y) \lesssim \left(\frac{\rho(x)}{t}\right)^{\delta_1}. \qquad (35)$$

Then

$$\frac{1}{\mu(B)} \int_B [g_2(f)(x)]^2\,d\mu(x) \lesssim \frac{1}{\mu(B)} \int_B \int_{8\rho(x)}^\infty \left(\frac{\rho(x)}{t}\right)^{2\delta_1} \frac{dt}{t}\,d\mu(x) \lesssim 1,$$

which together with (34) gives (31). Moreover, since (31) holds for all balls $B(x_0,\,r)$ with $r \geq \rho(x_0)$, we have that $g(f)(x) < \infty$ for a. e. $x \in \mathcal{X}$.

Now we assume that $B \equiv B(x_0,\,r)$ with $r < \rho(x_0)$. If $r \geq \rho(x_0)/8$, then by (1) and (31), we have

$$\frac{1}{\mu(B)} \int_B \left\{[g(f)(x)]^2 - \operatorname*{essinf}_B [g(f)]^2\right\} d\mu(x) \lesssim \frac{1}{\mu(8B)} \int_{8B} [g(f)(x)]^2\,d\mu(x) \lesssim 1,$$

which is desired. Assume that $r < \rho(x_0)/8$. It suffices to show that for μ-a. e. $y \in B$,

$$\frac{1}{\mu(B)} \int_B \left\{[g(f)(x)]^2 - [g(f)(y)]^2\right\} d\mu(x) \lesssim 1.$$

For all $x \in B$, write

$$[g(f)(x)]^2 = \int_0^{8r} |Q_t(f)(x)|^2 \frac{dt}{t} + \int_{8r}^{8\rho(x_0)} \cdots + \int_{8\rho(x_0)}^\infty \cdots$$

$$\equiv [g_r(f)(x)]^2 + [g_{r,\,x_0}(f)(x)]^2 + [g_\infty(f)(x)]^2.$$

Observe that for μ-a. e. $y \in B$,

$$\frac{1}{\mu(B)} \int_B \left\{[g(f)(x)]^2 - [g(f)(y)]^2\right\} d\mu(x)$$

$$\leq \frac{1}{\mu(B)} \int_B \left\{ [g_r(f)(x)]^2 + [g_\infty(f)(x)]^2 + [g_{r,\,x_0}(f)(x)]^2 - [g_{r,\,x_0}(f)(y)]^2 \right\} d\mu(x).$$

We first prove that

$$\frac{1}{\mu(B)} \int_B [g_r(f)(x)]^2 \, d\mu(x) \lesssim 1. \tag{36}$$

Write $f \equiv f_1 + f_2 + f_B$, where $f_1 \equiv (f - f_B)\chi_{2B}$ and $f_2 \equiv (f - f_B)\chi_{(2B)^{\complement}}$. By the $L^2(\mathcal{X})$-boundedness of g, (1) and Lemma 3.1, we have

$$\frac{1}{\mu(B)} \int_B [g_r(f_1)(x)]^2 \, d\mu(x) \lesssim \frac{1}{\mu(B)} \int_{2B} |f - f_B|^2 \, d\mu(x) \lesssim 1. \tag{37}$$

For all $x \in B$, by (Q)$_{\text{i}}$, (1) and the fact that $|f_{2^{j+1}B} - f_B| \lesssim j$ for all $j \in \mathbb{N}$, we have

$$
\begin{aligned}
|Q_t(f_2)(x)| &\lesssim \int_{(2B)^{\complement}} \frac{1}{V_t(x) + V(x, z)} \left(\frac{t}{t + d(x, z)} \right)^\gamma |f(z) - f_B| \, d\mu(z) \\
&\lesssim \sum_{j=1}^{\infty} \left(\frac{t}{2^{j-1}r} \right)^\gamma \left[\frac{1}{V_{2^{j-1}r}(x)} \int_{2^{j+1}B} [|f(z) - f_{2^{j+1}B}| + |f_{2^{j+1}B} - f_B|] \, d\mu(z) \right] \\
&\lesssim \left(\frac{t}{r} \right)^\gamma \sum_{j=1}^{\infty} j 2^{-j\gamma} \lesssim \left(\frac{t}{r} \right)^\gamma,
\end{aligned}
$$

which further implies that

$$\frac{1}{\mu(B)} \int_B [g_r(f_2)(x)]^2 \, d\mu(x) \lesssim \int_0^{8r} \left(\frac{t}{r} \right)^{2\gamma} \frac{dt}{t} \lesssim 1. \tag{38}$$

By (37) and (38), to prove (36), it remains to show that

$$\frac{1}{\mu(B)} \int_B [g_r(f_B)(x)]^2 \, d\mu(x) \lesssim 1. \tag{39}$$

Observe that $|f_B| \lesssim \log \frac{\rho(x_0)}{r}$. For all $x \in B$ and $t \in (0, \rho(x_0))$, from (Q)$_{\text{iii}}$ and the fact that $r < \rho(x_0)/8$ and (10), it follows that

$$|Q_t(f_B)(x)| \lesssim \left(\frac{t}{\rho(x)} \right)^{\delta_2} |f_B| \lesssim \left(\frac{t}{\rho(x_0)} \right)^{\delta_2} \log \frac{\rho(x_0)}{r},$$

which via $t \leq 8r < \rho(x_0)$ further yields (39).

Let $a \in [1/8, \infty)$ and $\widetilde{C}_a$ be as in (10). We now claim that for all $f \in \mathrm{BMO}_\rho(\mathcal{X})$ with $\|f\|_{\mathrm{BMO}_\rho(\mathcal{X})} = 1$, $x \in B$ and $t \leq 8\widetilde{C}_a \rho(x_0)$,

$$|Q_t(f)(x)| \lesssim 1. \tag{40}$$

In fact, by (Q)$_{\text{ii}}$, we obtain

$$|Q_t(f - f_{B(x, t)})(x)|$$

$$\lesssim \int_{\mathcal{X}} \frac{1}{V_t(x) + V(x, y)} \left(\frac{t}{t + d(x, y)} \right)^{\gamma} |f(y) - f_{B(x,t)}| \, d\mu(y)$$

$$\lesssim \sum_{j=0}^{\infty} 2^{-j\gamma} \frac{1}{V_{2^{j-1}t}(x)} \int_{d(x, y) < 2^j t} |f(y) - f_{B(x,t)}| \, d\mu(y) \lesssim 1. \tag{41}$$

Since $\delta_2 > 0$, by (Q)$_{\text{iii}}$ and the fact that for all $x \in \mathcal{X}$, $|f_{B(x,t)}| \lesssim 1 + \log \frac{\rho(x)}{t}$, we have

$$|Q_t(f_{B(x,t)})(x)| \lesssim \left(\frac{t}{\rho(x)} \right)^{\delta_2} \left(1 + \log \frac{\rho(x)}{t} \right) \lesssim 1.$$

Combining this and (41) proves the claim.

Using (40), (10) and (35), we have that for all $x \in B$,

$$\int_{8\rho(x_0)}^{\infty} |Q_t(f)(x)|^2 \frac{dt}{t} \leq \int_{8\widetilde{C}_a\rho(x_0)}^{\infty} |Q_t(f)(x)|^2 \frac{dt}{t} + \int_{8\rho(x_0)}^{8\widetilde{C}_a\rho(x_0)} \cdots$$

$$\lesssim \int_{8\widetilde{C}_a\rho(x_0)}^{\infty} \left(\frac{\rho(x)}{t} \right)^{2\delta_1} \frac{dt}{t} + 1 \lesssim 1,$$

which gives that

$$\frac{1}{\mu(B)} \int_B [g_{\infty}(f)(x)]^2 \, d\mu(x) \lesssim 1. \tag{42}$$

By (36) and (42), to finish the proof of Theorem 6.1, it remains to show that for μ-a. e. $y \in B$,

$$\frac{1}{\mu(B)} \int_B \left\{ [g_{r, x_0}(f)(x)]^2 - [g_{r, x_0}(f)(y)]^2 \right\} \, d\mu(x) \lesssim 1. \tag{43}$$

From (40), we deduce that for μ-a. e. $x, y \in B$,

$$\left\{ [g_{r, x_0}(f)(x)]^2 - [g_{r, x_0}(f)(y)]^2 \right\}$$

$$\leq \int_{8r}^{8\rho(x_0)} |Q_t(f)(x) + Q_t(f)(y)| |Q_t(f)(x) - Q_t(f)(y)| \frac{dt}{t}$$

$$\lesssim \int_{8r}^{8\rho(x_0)} |Q_t(f)(x) - Q_t(f)(y)| \frac{dt}{t}.$$

For $t \in (8r, 8\rho(x_0))$ and $x, y \in B$, we write

$$|Q_t(f)(x) - Q_t(f)(y)|$$

$$\leq \left| \int_{\mathcal{X}} [Q_t(x, z) - Q_t(y, z)] [f(z) - f_B] \, d\mu(z) \right| + |f_B| \left| \int_{\mathcal{X}} [Q_t(x, z) - Q_t(y, z)] \, d\mu(z) \right|$$

$$\equiv J_1 + J_2.$$

By (Q)$_{\text{ii}}$, $t \in (8r, 8\rho(x_0))$, (1) and the fact that $2^{j+1}B \subset B(x, 2^{j+2}r)$ for all $x \in B$, we obtain

$$J_1 \lesssim \int_{\mathcal{X}} \left(\frac{d(x, y)}{t + d(x, z)} \right)^{\beta} \frac{1}{V_t(x) + V(x, z)} \left(\frac{t}{t + d(x, z)} \right)^{\gamma} |f(z) - f_B| \, d\mu(z)$$

$$\lesssim \sum_{j=0}^{\infty} \frac{r^\beta t^\gamma}{(t+2^{j-1}r)^{\beta+\gamma}} \left\{ \frac{1}{V_{2^{j-1}r}(x)} \int_{2^{j+1}B} |f(z) - f_{2^{j+1}B}| \, d\mu(z) + |f_{2^{j+1}B} - f_B| \right\}$$

$$\lesssim \sum_{j=0}^{\infty} \frac{r^\beta t^\gamma}{(t+2^j r)^{\beta+\gamma}} (j+2).$$

From this, we deduce that

$$\int_{8r}^{8\rho(x_0)} J_1 \frac{dt}{t} \lesssim r^\beta \sum_{j=0}^{\infty} (j+2) \int_{8r}^{8\rho(x_0)} \frac{t^{\gamma-1}}{(t+2^j r)^{\beta+\gamma}} \, dt \lesssim 1. \tag{44}$$

On the other hand, we obtain that for μ-a. e. $x, y \in B$,

$$\left| \int_{\mathcal{X}} [Q_t(x,z) - Q_t(y,z)] \, d\mu(z) \right|$$

$$\lesssim \int_{\mathcal{X}} \left(\frac{d(x,y)}{t+d(x,z)} \right)^\beta \frac{1}{V_t(x) + V(x,z)} \left(\frac{t}{t+d(x,z)} \right)^\gamma d\mu(z)$$

$$\lesssim \sum_{j=0}^{\infty} \left(\frac{r}{2^j t} \right)^\beta 2^{-j\gamma} \lesssim \left(\frac{r}{t} \right)^\beta.$$

The fact that $r < \rho(x_0)/8$ together with (10) and $|f_B| \lesssim \log \frac{\rho(x_0)}{r}$ implies that for μ-a. e. x, $y \in B$,

$$\int_{8r}^{8\rho(x_0)} J_2 \frac{dt}{t} \leq \int_{8r}^{8\rho(x_0)} \log \frac{\rho(x_0)}{r} \left| \int_{\mathcal{X}} [Q_t(x,z) - Q_t(y,z)] \, d\mu(z) \right|^{\frac{2}{3}} \left(\frac{t}{\rho(x_0)} \right)^{\frac{\delta_2}{3}} \frac{dt}{t}$$

$$\lesssim \int_{8r}^{8\rho(x_0)} \log \frac{\rho(x_0)}{r} \left(\frac{r}{t} \right)^{\frac{\beta}{3}} \left(\frac{r}{\rho(x_0)} \right)^{\min\left(\frac{\beta}{3}, \frac{\delta_2}{3} \right)} \frac{dt}{t} \lesssim \int_{8r}^{\infty} \left(\frac{r}{t} \right)^{\frac{\beta}{3}} \frac{dt}{t} \lesssim 1.$$

This together with (44) leads to (43), and hence, finishes the proof of Theorem 6.1. $\square$

As a consequence of Theorem 6.1, we have the following conclusion.

Corollary 6.1. *With the assumptions same as in Theorem 6.1, then there exists a positive constant C such that for all $f \in \mathrm{BMO}_\rho(\mathcal{X})$, $g(f) \in \mathrm{BLO}_\rho(\mathcal{X})$ and $\|g(f)\|_{\mathrm{BLO}_\rho(\mathcal{X})} \leq C\|f\|_{\mathrm{BMO}_\rho(\mathcal{X})}$.*

Proof. Since

$$g(f) - \operatorname*{essinf}_B g(f) \leq \left\{ [g(f)]^2 - \operatorname*{essinf}_B [g(f)]^2 \right\}^{1/2},$$

by the Hölder inequality and Theorem 6.1, we have that for all balls $B \notin \mathcal{D}$,

$$\frac{1}{\mu(B)} \int_B \left[g(f)(x) - \operatorname*{essinf}_B g(f) \right] d\mu(x)$$

$$\lesssim \frac{1}{\mu(B)} \int_B \left\{ [g(f)(x)]^2 - \operatorname*{essinf}_B [g(f)]^2 \right\}^{1/2} d\mu(x)$$

$$\lesssim \left\{ \frac{1}{\mu(B)} \int_B \left\{ [g(f)(x)]^2 - \operatorname*{essinf}_B [g(f)]^2 \right\} d\mu(x) \right\}^{1/2} \lesssim \|f\|_{\mathrm{BMO}_\rho(\mathcal{X})}.$$

On the other hand, by (31) and the Hölder inequality, we obtain that for all balls $B \in \mathcal{D}$,

$$\frac{1}{\mu(B)} \int_B g(f)(x) \, d\mu(x) \leq \left(\frac{1}{\mu(B)} \int_B [g(f)(x)]^2 \, d\mu(x) \right)^{1/2} \lesssim \|f\|_{\mathrm{BMO}_\rho(\mathcal{X})}.$$

Combining the two inequalities above finishes the proof of Corollary 6.1.   □

7. **Applications.** This section is divided into Subsections 7.1 through 7.4, which are devoted to the applications of results obtained in Sections 5 and 6, respectively, to the Schrödinger operator or the degenerate Schrödinger operator on $\mathbb{R}^d$, the sub-Laplace Schrödinger operator on Heisenberg groups or on connected and simply connected nilpotent Lie groups.

7.1. **Schrödinger operators on $\mathbb{R}^d$.** Let d be a positive integer and $d \geq 3$, and $\mathbb{R}^d$ be the d-dimensional Euclidean space endowed with the Euclidean norm $|\cdot|$ and the Lebesgue measure dx. Denote the Laplacian $\sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$ on $\mathbb{R}^d$ by Δ and the corresponding heat (Gauss) semigroup $\{e^{t\Delta}\}_{t>0}$ by $\{\widetilde{T}_t\}_{t>0}$. Let V be a nonnegative locally integrable function on $\mathbb{R}^d$, $\mathcal{L} \equiv -\Delta + V$ be the Schrödinger operator and $\{T_t\}_{t>0}$ be the corresponding semigroup. Moreover, for all $t > 0$ and $x, y \in \mathbb{R}^d$, set

$$Q_t(x, y) \equiv t^2 \left. \frac{dT_s(x, y)}{ds} \right|_{s=t^2}.$$

Let $q \in (d/2, d]$, $V \in \mathcal{B}_q(\mathbb{R}^d, |\cdot|, dx)$ and ρ be as in (4). Then we have the following estimates; see [8, 6, 7].

Proposition 7.1. *Let $q \in (d/2, d]$, $\beta \in (0, 2 - d/q)$ and $N \in \mathbb{N}$. Then there exist positive constants $\widetilde{C}$ and C, where C is independent of N, such that for all $t \in (0, \infty)$ and $x, x', y \in \mathcal{X}$ with $d(x, x') \leq \sqrt{t}/2$,*

   (i) $|T_t(x, y)| \leq \widetilde{C} t^{-d/2} \exp\{-\frac{|x-y|^2}{Ct}\} [\frac{\rho(x)}{\sqrt{t}+\rho(x)}]^N [\frac{\rho(y)}{\sqrt{t}+\rho(y)}]^N$;

   (ii) $|T_t(x, y) - T_t(x', y)| \leq \widetilde{C} [\frac{|x-x'|}{\sqrt{t}}]^\beta t^{-d/2} \exp\{-\frac{|x-y|^2}{Ct}\} [\frac{\rho(x)}{\sqrt{t}+\rho(x)}]^N [\frac{\rho(y)}{\sqrt{t}+\rho(y)}]^N$;

   (iii) $|T_t(x, y) - \widetilde{T}_t(x, y)| \leq \widetilde{C} [\frac{\sqrt{t}}{\sqrt{t}+\rho(x)}]^{2-d/q} t^{-d/2} \exp\{-\frac{|x-y|^2}{Ct}\}$,

and for all $t \in (0, \infty)$ and $x, x', y \in \mathcal{X}$ with $d(x, x') \leq t/2$,

   (iv) $|Q_t(x, y)| \leq \widetilde{C} t^{-d} \exp\{-\frac{|x-y|^2}{Ct^2}\} [\frac{\rho(x)}{t+\rho(x)}]^N [\frac{\rho(y)}{t+\rho(y)}]^N$;

   (v) $|Q_t(x, y) - Q_t(x', y)| \leq \widetilde{C} [\frac{|x-x'|}{t}]^\beta t^{-d} \exp\{-\frac{|x-y|^2}{Ct^2}\} [\frac{\rho(x)}{t+\rho(x)}]^N [\frac{\rho(y)}{t+\rho(y)}]^N$;

   (vi) $|\int_{\mathbb{R}^d} Q_t(x, y) d\mu(y)| \leq \widetilde{C} [\frac{t}{\rho(x)}]^{2-d/q} [\frac{\rho(x)}{t+\rho(x)}]^N$.

Observe that $\{\widetilde{T}_{t^2}\}_{t>0}$ is a continuous $(1, N)$-AOTI for all positive constants N. Thus $\{T_{t^2}\}_{t>0}$ and $\{\widetilde{T}_{t^2}\}_{t>0}$ satisfy the assumption (22). Moreover, the $L^2(\mathbb{R}^d)$-boundedness of g-function was obtained in [8]. Using these facts and Proposition 7.1 and applying Theorems 5.1, 5.2 and 6.1, and Corollaries 5.1, 5.2 and 6.1, we have the following result.

Proposition 7.2. *Let $q \in (d/2, \infty]$, $V \in \mathcal{B}_q(\mathbb{R}^d, |\cdot|, dx)$ and ρ be as in (4). There exists a positive constant C such that for all $f \in \mathrm{BMO}_\rho(\mathbb{R}^d)$, $T^+(f)$, $\widetilde{T}^+(f)$, $\widetilde{T}_\rho^+(f)$, $P^+(f)$, $\widetilde{P}^+(f)$, $\widetilde{P}_\rho^+(f)$, $g(f)$, $[g(f)]^2 \in \mathrm{BLO}_\rho(\mathbb{R}^d)$ and*

$$\|T^+(f)\|_{\mathrm{BLO}_\rho(\mathbb{R}^d)} + \|\widetilde{T}^+(f)\|_{\mathrm{BLO}_\rho(\mathbb{R}^d)} + \|T_\rho^+(f)\|_{\mathrm{BLO}_\rho(\mathbb{R}^d)} + \|P^+(f)\|_{\mathrm{BLO}_\rho(\mathbb{R}^d)}$$
$$+ \|\widetilde{P}^+(f)\|_{\mathrm{BLO}_\rho(\mathbb{R}^d)} + \|\widetilde{P}_\rho^+(f)\|_{\mathrm{BLO}_\rho(\mathbb{R}^d)} + \|g(f)\|_{\mathrm{BLO}_\rho(\mathbb{R}^d)} + \|[g(f)]^2\|_{\mathrm{BLO}_\rho(\mathbb{R}^d)}^{1/2}$$
$$\leq C\|f\|_{\mathrm{BMO}_\rho(\mathbb{R}^d)}.$$

We also point out that when ρ is as in (4), Dziubański et al [9] obtained the boundedness of T^+, P^+ and g on $\mathrm{BMO}_\rho(\mathbb{R}^d)$. Proposition 7.2 improves their results.

7.2. Degenerate Schrödinger operators on $\mathbb{R}^d$. Let $d \geq 3$ and $\mathbb{R}^d$ be the d-dimensional Euclidean space endowed with the Euclidean norm $|\cdot|$ and the Lebesgue measure dx. Recall that a nonnegative locally integrable function w is said to be an $A_2(\mathbb{R}^d)$ weight in the sense of Muckenhoupt if

$$\sup_{B \subset \mathbb{R}^d} \left\{ \frac{1}{|B|} \int_B w(x)\, dx \right\}^{1/2} \left\{ \frac{1}{|B|} \int_B [w(x)]^{-1}\, dx \right\}^{1/2} < \infty,$$

where the supremum is taken over all the balls in $\mathbb{R}^d$. Observe that if we set $w(E) \equiv \int_E w(x)dx$ for any measurable set E, then there exist positive constants C, Q and κ such that for all $x \in \mathbb{R}^d$, $\lambda > 1$ and $r > 0$,

$$C^{-1}\lambda^\kappa w(B(x, r)) \leq w(B(x, \lambda r)) \leq C\lambda^Q w(B(x, r)),$$

namely, the measure $w(x)\, dx$ satisfies (1). Thus $(\mathbb{R}^d, |\cdot|, w(x)\, dx)$ is an RD-space.

Let $w \in A_2(\mathbb{R}^d)$ and $\{a_{i,j}\}_{1 \leq i, j \leq d}$ be a real symmetric matrix function satisfying that for all $x, \xi \in \mathbb{R}^d$,

$$C^{-1}|\xi|^2 \leq \sum_{1 \leq i, j \leq d} a_{i,j}(x)\xi_i \overline{\xi}_j \leq C|\xi|^2.$$

Then the degenerate elliptic operator $\mathcal{L}_0$ is defined by

$$\mathcal{L}_0 f(x) \equiv -\frac{1}{w(x)} \sum_{1 \leq i, j \leq d} \partial_i (a_{i,j}(\cdot)\partial_j f)(x),$$

where $x \in \mathbb{R}^d$. Denote by $\{\widetilde{T}_t\}_{t>0} \equiv \{e^{-t\mathcal{L}_0}\}_{t>0}$ the semigroup generated by $\mathcal{L}_0$. We also denote the kernel of $\widetilde{T}_t$ by $\widetilde{T}_t(x, y)$ for all $x, y \in \mathbb{R}^d$ and $t \in (0, \infty)$. Then it is known that there exist positive constants C, C_6, $\widetilde{C}_6$ and $\alpha \in (0, 1]$ such that for all $t \in (0, \infty)$ and $x, y \in \mathbb{R}^d$,

$$C^{-1}\frac{1}{V_{\sqrt{t}}(x)} \exp\left\{ -\frac{|x-y|^2}{\widetilde{C}_6 t} \right\} \leq \widetilde{T}_t(x, y) \leq C\frac{1}{V_{\sqrt{t}}(x)} \exp\left\{ -\frac{|x-y|^2}{C_6 t} \right\};$$

that for all $t \in (0, \infty)$ and $x, y, y' \in \mathbb{R}^d$ with $|y - y'| < |x - y|/4$,

$$|\widetilde{T}_t(x, y) - \widetilde{T}_t(x, y')| \leq C \frac{1}{V_{\sqrt{t}}(x)} \left(\frac{|y - y'|}{\sqrt{t}} \right)^\alpha \exp\left\{ -\frac{|x - y|^2}{C_6 t} \right\};$$

and that for all $t \in (0, \infty)$ and $x, y \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} \widetilde{T}_t(x, z) \, w(z) \, dz = 1 = \int_{\mathbb{R}^d} \widetilde{T}_t(z, y) \, w(z) \, dz;$$

see, for example, Theorems 2.1, 2.7, 2.3, 2.4 and Corollary 3.4 of [17].

Let V be a nonnegative locally integrable function on $w(x) \, dx$. Define the degenerate Schrödinger operator by $\mathcal{L} \equiv \mathcal{L}_0 + V$. Then $\mathcal{L}$ generates a semigroup $\{T_t\}_{t>0} \equiv \{e^{-t\mathcal{L}}\}_{t>0}$ with kernels $\{T_t(x, y)\}_{t>0}$. Moreover, for all $t \in (0, \infty)$ and $x, y \in \mathbb{R}^d$, set

$$Q_t(x, y) \equiv t^2 \frac{dT_s(x, y)}{ds}\bigg|_{s=t^2}.$$

Let $q \in (Q/2, Q]$, $V \in \mathcal{B}_q(\mathbb{R}^d, |\cdot|, w(x) \, dx)$ and ρ be as in (4). Then $\{T_t\}_{t>0}$ and $\{Q_t\}_{t>0}$ satisfy Proposition 7.1 with $t^{-d/2}$ replaced by $V_{\sqrt{t}}(x)$, t^{-d} by $V_t(x)$, and d by Q.

In fact, the corresponding Proposition 7.1 (i) and (iii) here were given in [8]. The proof of (ii) here is similar to that of Proposition 7.1; see [7] and also Lemma 7.4 below. The proofs of the corresponding Proposition 7.1 (iv), (v) and (vi) here are similar to that of Proposition 4 of [9]. We omit the details here.

Observe that $\{\widetilde{T}_{t^2}\}_{t>0}$ is a continuous $(1, N)$-AOTI for all positive constants N. Thus $\{T_{t^2}\}_{t>0}$ and $\{\widetilde{T}_{t^2}\}_{t>0}$ satisfy the assumption (22). Moreover, the $L^2(\mathbb{R}^d)$-boundedness of g-function can be obtained by the same argument as in Lemma 3 of [8]. Using these facts and applying Theorems 5.1, 5.2 and 6.1, and Corollaries 5.1, 5.2 and 6.1, we have the following conclusions.

Proposition 7.3. *Let $w \in A_2(\mathbb{R}^d)$. Let $q \in (Q/2, \infty]$, $V \in \mathcal{B}_q(\mathbb{R}^d, |\cdot|, w(x) \, dx)$ and ρ be as in (4) with $d\mu = w(x) \, dx$. Then there exists a positive constant C such that for all $f \in \mathrm{BMO}_\rho(w(x) \, dx)$, $T^+(f)$, $\widetilde{T}^+(f)$, $\widetilde{T}^+_\rho(f)$, $P^+(f)$, $\widetilde{P}^+(f)$, $\widetilde{P}^+_\rho(f)$, $g(f)$, $[g(f)]^2 \in \mathrm{BLO}_\rho(w(x) \, dx)$ and*

$$\|T^+(f)\|_{\mathrm{BLO}_\rho(w(x) \, dx)} + \|\widetilde{T}^+(f)\|_{\mathrm{BLO}_\rho(w(x) \, dx)} + \|T^+_\rho(f)\|_{\mathrm{BLO}_\rho(w(x) \, dx)}$$

$$+ \|P^+(f)\|_{\mathrm{BLO}_\rho(w(x) \, dx)} + \|\widetilde{P}^+(f)\|_{\mathrm{BLO}_\rho(w(x) \, dx)} + \|\widetilde{P}^+_\rho(f)\|_{\mathrm{BLO}_\rho(w(x) \, dx)}$$

$$+ \|g(f)\|_{\mathrm{BLO}_\rho(w(x) \, dx)} + \|[g(f)]^2\|_{\mathrm{BLO}_\rho(w(x) \, dx)}^{1/2}$$

$$\leq C\|f\|_{\mathrm{BMO}_\rho(w(x) \, dx)}.$$

7.3. Schrödinger operators on Heisenberg groups.
The $(2n+1)$-dimensional Heisenberg group $\mathbb{H}^n$ is a connected and simply connected nilpotent Lie groups with the underlying manifold $\mathbb{R}^{2n} \times \mathbb{R}$ and the multiplication

$$(x, s)(y, s) = \left(x + y, \, t + s + 2 \sum_{j=1}^n [x_{n+j} y_j - x_j y_{n+j}] \right).$$

The homogeneous norm on $\mathbb{H}^n$ is defined by $|(x, t)| = (|x|^4 + |t|^2)^{1/4}$ for all $(x, t) \in \mathbb{H}^n$, which induces a left-invariant metric $d((x, t), (y, s)) = |(-x, -t)(y, s)|$. Moreover, there exists a positive constant C such that $|B((x, t), r)| = Cr^Q$, where $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$ and $|B((x, t), r)|$ is the Lebesgue measure of the ball $B((x, t), r)$. The triplet $(\mathbb{H}^n, d, dx)$ is an RD-space.

A basis for the Lie algebra of left invariant vector fields on $\mathbb{H}^n$ is given by

$$X_{2n+1} = \frac{\partial}{\partial t}, \quad X_j = \frac{\partial}{\partial x_j} + 2x_{n+j}\frac{\partial}{\partial t}, \quad X_{n+j} = \frac{\partial}{\partial x_{n+j}} - 2x_j\frac{\partial}{\partial t}, \; j = 1, \cdots, n.$$

All non-trivial commutators are $[X_j, X_{n+j}] = -4X_{2n+1}, j = 1, \cdots, n$. The sub-Laplacian has the form $\Delta_{\mathbb{H}^n} = \sum_{j=1}^{2n} X_j^2$.

Let V be a nonnegative locally integrable function on $\mathbb{H}^n$. Define the sub-Laplacian Schrödinger operator by $\mathcal{L} \equiv -\Delta_{\mathbb{H}^n} + V$. Denote by $\{T_t\}_{t>0} \equiv \{e^{-t\mathcal{L}}\}_{t>0}$ the semigroup generated by $\mathcal{L}$ and by $\{\widetilde{T}_t\}_{t>0} \equiv \{e^{t\Delta_{\mathbb{H}^n}}\}_{t>0}$ the semigroup generated by $-\Delta_{\mathbb{H}^n}$.

Let $V \in \mathcal{B}_q(\mathbb{H}^n, d, dx)$ with $q \in (n+1, 2n+2]$ and ρ be as in (4). Then $\{T_t\}_{t>0}$ and $\{Q_t\}_{t>0}$ satisfy Proposition 7.1 with d replaced by $2(n+2)$ and $|x-y|$ replaced by $d(x, y)$; see [22].

Observe that $\{\widetilde{T}_{t^2}\}_{t>0}$ is a continuous $(1, N)$-AOTI for all positive constants N. Thus $\{T_{t^2}\}_{t>0}$ and $\{\widetilde{T}_{t^2}\}_{t>0}$ satisfy the assumption (22). Moreover, the $L^2(\mathbb{H}^n)$-boundedness of g-function was obtained in [22]. Using these facts and applying Theorems 5.1, 5.2 and 6.1, and Corollaries 5.1, 5.2 and 6.1, we have the following conclusions.

Proposition 7.4. *Let $q \in (n+1, \infty]$, $V \in \mathcal{B}_q(\mathbb{H}^n, d, dx)$ and ρ be as in (4). Then there exists a positive constant C such that for all $f \in \mathrm{BMO}_\rho(\mathbb{H}^n)$, $T^+(f)$, $\widetilde{T}^+(f)$, $\widetilde{T}_\rho^+(f)$, $P^+(f)$, $\widetilde{P}^+(f)$, $\widetilde{P}_\rho^+(f)$, $g(f)$, $[g(f)]^2 \in \mathrm{BLO}_\rho(\mathbb{H}^n)$ and*

$$\|T^+(f)\|_{\mathrm{BLO}_\rho(\mathbb{H}^n)} + \|\widetilde{T}^+(f)\|_{\mathrm{BLO}_\rho(\mathbb{H}^n)} + \|T_\rho^+(f)\|_{\mathrm{BLO}_\rho(\mathbb{H}^n)} + \|P^+(f)\|_{\mathrm{BLO}_\rho(\mathbb{H}^n)}$$
$$+ \|\widetilde{P}^+(f)\|_{\mathrm{BLO}_\rho(\mathbb{H}^n)} + \|\widetilde{P}_\rho^+(f)\|_{\mathrm{BLO}_\rho(\mathbb{H}^n)} + \|g(f)\|_{\mathrm{BLO}_\rho(\mathbb{H}^n)} + \|[g(f)]^2\|_{\mathrm{BLO}_\rho(\mathbb{H}^n)}^{1/2}$$
$$\leq C\|f\|_{\mathrm{BMO}_\rho(\mathbb{H}^n)}.$$

We also point out that when ρ is as in (4), Lin and Liu [22] introduced $\mathrm{BMO}_\rho(\mathbb{H}^n)$ and established the boundedness of T^+, P^+ and g on $\mathrm{BMO}_\rho(\mathbb{H}^n)$. The results in this subsection improve their corresponding results.

7.4. Schrödinger operators on connected and simply connected nilpotent Lie groups.
Let $\mathbb{G}$ be a connected and simply connected nilpotent Lie group. Let $X \equiv \{X_1, \cdots, X_k\}$ be left invariant vector fields on $\mathbb{G}$ satisfying the Hörmander condition that $\{X_1, \cdots, X_k\}$ together with their commutators of order $\leq m$ generates the tangent space of $\mathbb{G}$ at each point of $\mathbb{G}$. Let d be the Carnot-Carathéodory (control) distance on $\mathbb{G}$ associated to $\{X_1, \cdots, X_k\}$. Fix a left invariant Haar measure μ on $\mathbb{G}$. Then for all $x \in \mathbb{G}$, $V_r(x) = V_r(e)$; moreover, there exist $\kappa, D \in (0, \infty)$ with $\kappa \leq D$ such that for all $x \in \mathbb{G}$,

$$C^{-1}r^\kappa \leq V_r(x) \leq Cr^\kappa \tag{45}$$

when $r \in (0, 1]$, and $C^{-1} r^D \leq V_r(x) \leq C r^D$ when $r \in (1, \infty)$; see [25] and [29]. Thus $(\mathbb{G}, d, \mu)$ is an RD-space.

The sub-Laplacian is given by $\Delta_{\mathbb{G}} \equiv \sum_{j=1}^{k} X_j^2$. Denote by $\{\widetilde{T}_t\}_{t>0} \equiv \{e^{t\Delta_{\mathbb{G}}}\}_{t>0}$ the semigroup generated by $-\Delta_{\mathbb{G}}$. Then there exist positive constants C, C_7 and $\widetilde{C}_7$ such that for all $t \in (0, \infty)$ and x, $y \in \mathbb{G}$,

$$C^{-1} \frac{1}{V_{\sqrt{t}}(x)} \exp\left\{-\frac{[d(x, y)]^2}{\widetilde{C}_7 t}\right\} \leq \widetilde{T}_t(x, y) \leq C \frac{1}{V_{\sqrt{t}}(x)} \exp\left\{-\frac{[d(x, y)]^2}{C_7 t}\right\}, \qquad (46)$$

that for all $t \in (0, \infty)$ and x, y, $y' \in \mathbb{G}$ with $d(y, y') \leq d(x, y)/4$,

$$|\widetilde{T}_t(x, y) - \widetilde{T}_t(x, y')| \leq C \frac{d(y, y')}{\sqrt{t}} \frac{1}{V_{\sqrt{t}}(x)} \exp\left\{-\frac{[d(x, y)]^2}{C_7 t}\right\}, \qquad (47)$$

and that for all $t \in (0, \infty)$ and x, $y \in \mathbb{G}$,

$$\int_{\mathbb{G}} \widetilde{T}_t(x, z) \, d\mu(z) = 1 = \int_{\mathbb{G}} \widetilde{T}_t(z, y) \, d\mu(z);$$

see, for example, [29].

Define the radial maximal operator $\widetilde{T}^+$ by $\widetilde{T}^+(f)(x) \equiv \sup_{t>0} |\widetilde{T}_t(f)(x)|$ for all $x \in \mathbb{G}$. Then by (46), it is easy to see that $\widetilde{T}^+$ is bounded on $L^p(\mathbb{G})$ for $p \in (1, \infty]$.

Let V be a nonnegative locally integrable function on $\mathbb{G}$. Then the sub-Laplace Schrödinger operator $\mathcal{L}$ is defined by $\mathcal{L} \equiv -\Delta_{\mathbb{G}} + V$. The operator $\mathcal{L}$ generates a semigroup of operators $\{T_t\}_{t>0} \equiv \{e^{-t\mathcal{L}}\}_{t>0}$, whose kernels are denoted by $\{T_t(x, y)\}_{t>0}$. Define the radial maximal operator T^+ by $T^+(f)(x) \equiv \sup_{t>0} |e^{-t\mathcal{L}}(f)(x)|$ for all $x \in \mathbb{G}$. Then from Lemma 7.1 below, it is easy to see that T^+ is bounded on $L^p(\mathbb{G})$ for $p \in (1, \infty]$.

Let $q > D/2$, $V \in \mathcal{B}_q(\mathbb{G}, d, \mu)$ and ρ be as in (4). Then Li [21] established some basic results concerning $\mathcal{L}$, which include estimates for fundamental solutions of $\mathcal{L}$ and the boundedness on Lebesgue spaces of some operators associated to $\mathcal{L}$. To apply the results obtained in Sections 5 and 6 to $\mathcal{L}$, we need the following estimate, which is a consequence of Proposition 5.2 and (5.12) in [31] together with the symmetry of T_t and the fact that for all x, $y \in \mathbb{G}$ and $t \in (0, \infty)$, $V_t(x) \sim V_t(y)$. We omit the details.

Lemma 7.1. *Let $q \in (D/2, D]$ and $V \in \mathcal{B}_q(\mathbb{G}, d, \mu)$. Then for all $N \in (0, \infty)$, there exist positive constants C and C_8, where C_8 is independent of N, such that for all $t \in (0, \infty)$ and x, $y \in \mathbb{G}$,*

$$0 \leq T_t(x, y) \leq C \frac{1}{V_{\sqrt{t}}(x)} \exp\left\{-\frac{[d(x, y)]^2}{C_8 t}\right\} \left[\frac{\rho(x)}{\rho(x) + \sqrt{t}}\right]^N \left[\frac{\rho(y)}{\rho(y) + \sqrt{t}}\right]^N.$$

For $t \in [0, \infty)$, set $E_t \equiv \widetilde{T}_t - T_t$. Denote also by E_t the kernel of E_t. The following estimate for E_t was established in [31].

Lemma 7.2. *If $q \in (D/2, D]$ and $V \in \mathcal{B}_q(\mathbb{G}, d, \mu)$, then for all $N \in (0, \infty)$, there exist positive constants C and C_9, where C_9 is independent of N, such that for all $t \in (0, \infty)$ and $x, y \in \mathbb{G}$,*

$$0 \le E_t(x, y) \le C \left[\frac{\sqrt{t}}{\sqrt{t} + \rho(x)} \right]^{2 - D/q} \frac{1}{V_{\sqrt{t}}(x)} \exp \left\{ -\frac{[d(x, y)]^2}{C_9 t} \right\}.$$

Moreover, to estimate the regularity of T_t, we need the regularity of E_t. To this end, we recall the following lemma.

Lemma 7.3. *If $q \in (D/2, D]$ and $V \in \mathcal{B}_q(\mathbb{G}, d, \mu)$, then for all positive constants C and $\widetilde{C}$, there exists positive constant A_6 such that for all $x \in \mathbb{G}$ and $t > 0$, when $\sqrt{t} < C\rho(x)$,*

$$\int_{\mathbb{G}} \frac{V(z)}{V_{\sqrt{t}}(x)} \exp \left\{ -\frac{[d(x, z)]^2}{\widetilde{C} t} \right\} d\mu(z) \le A_6 \frac{1}{t} \left[\frac{\sqrt{t}}{\rho(x)} \right]^{2 - D/q},$$

while when $\sqrt{t} \ge C\rho(x)$,

$$\int_{\mathbb{G}} \frac{V(z)}{V_{\sqrt{t}}(x)} \exp \left\{ -\frac{[d(x, z)]^2}{\widetilde{C} t} \right\} d\mu(z) \le A_6 \frac{1}{t} \left[\frac{\sqrt{t}}{\rho(x)} \right]^{\ell_0},$$

where ℓ_0 is a positive constant independent of $C, \widetilde{C}$ and A_6.

We remark that Lemma 7.3 with $\sqrt{t} < C\rho(x)$ is just Lemma 5.1 of [31]. For $\sqrt{t} \ge C\rho(x)$, the result can be proved similarly. We omit the details.

Lemma 7.4. *If $q \in (D/2, D]$ and $V \in \mathcal{B}_q(\mathbb{G}, d, \mu)$, then for all $\delta' \in (0, 2 - D/q)$, there exist positive constants C and A_7, where A_7 is independent of δ, such that for all $t \in (0, \infty)$ and $x', x, y \in \mathbb{G}$ with $d(x, x') < \min\{d(x, y)/4, \rho(x)\}$,*

$$|E_t(x, y) - E_t(x', y)| \le C \left[\frac{d(x, x')}{\rho(y)} \right]^{\delta'} \frac{1}{V_{\sqrt{t}}(x)} \exp \left\{ -\frac{[d(x, y)]^2}{A_7 t} \right\}.$$

Proof. Let $x', x, y \in \mathbb{G}$ with $d(x, x') < \min\{d(x, y)/4, \rho(x)\}$. Notice that if $d(x, x') < d(x, y)/4$, then $d(x, y) \sim d(x', y)$. We first prove that for all $\delta' \in (0, 2 - D/q)$ and $x, y \in \mathbb{G}$,

$$|E_t(x, y) - E_t(x', y)| \lesssim \left[\frac{d(x, x')}{\rho(y)} \right]^{\delta'} \frac{1}{V_{\sqrt{t}}(x)}. \tag{48}$$

If $d(x, x') \ge \rho(y)$, then (48) follows from Lemma 7.2. If $d(x, x') < \rho(y)$ and $t \le 2[d(x, x')]^2$, another application of Lemma 7.2 together with the symmetry of T_t and $\widetilde{T}_t$ also yields (48). Thus we may assume that $d(x, x') < \rho(y)$ and $t > 2[d(x, x')]^2$.

Recall (see, for example, [31, 8]) that for all $x, y \in \mathbb{G}$,

$$E_t(x, y) = \widetilde{T}_t(x, y) - T_t(x, y) = \int_0^t \int_{\mathbb{G}} \widetilde{T}_{t-s}(x, z) V(z) T_s(z, y) \, d\mu(z) \, ds.$$

We write

$$|E_t(x,\,y) - E_t(x',\,y)| \le \int_0^t \int_{\mathbb{G}} |\widetilde{T}_{t-s}(x,\,z) - \widetilde{T}_{t-s}(x',\,z)| V(z) T_s(z,\,y)\, d\mu(z)\, ds$$

$$= \int_0^{t/2} \int_{\mathbb{G}} |\widetilde{T}_{t-s}(x,\,z) - \widetilde{T}_{t-s}(x',\,z)| V(z) T_s(z,\,y)\, d\mu(z)\, ds$$

$$+ \int_0^{t/2} \int_{\mathbb{G}} |\widetilde{T}_s(x,\,z) - \widetilde{T}_s(x',\,z)| V(z) T_{t-s}(z,\,y)\, d\mu(z)\, ds$$

$$\equiv \mathrm{F}_1 + \mathrm{F}_2.$$

To estimate F_1, we consider the following two cases. Case (i) $t < 2[\rho(y)]^2$. For $s \in (0, t/2)$, we have $t - s \sim t$. By (47), Lemma 7.1, Lemma 7.3, the symmetry of $\widetilde{T}_t$, the assumption that $D/2 < q \le D$ and the fact that $V_r(x) \sim V_r(y)$ for all $x,\,y \in \mathbb{G}$ and $r \in (0, \infty)$, we have

$$\mathrm{F}_1 \lesssim \frac{d(x, x')}{\sqrt{t}} \frac{1}{V_{\sqrt{t}}(x)} \int_0^{t/2} \int_{\mathbb{G}} V(z) \frac{1}{V_{\sqrt{s}}(y)} \exp\left\{ -\frac{[d(z,\,y)]^2}{Cs} \right\} d\mu(z)\, ds$$

$$\lesssim \frac{d(x, x')}{\sqrt{t}} \frac{1}{V_{\sqrt{t}}(x)} \int_0^{t/2} \frac{1}{s} \left[\frac{\sqrt{s}}{\rho(y)} \right]^{2-D/q} ds \lesssim \left[\frac{d(x, x')}{\rho(y)} \right]^{2-D/q} \frac{1}{V_{\sqrt{t}}(x)}.$$

Case (ii) $t \ge 2[\rho(y)]^2$. Let ℓ_0 be as in Lemma 7.3 and $N > \ell_0$. Using (47), Lemma 7.1 and Lemma 7.3, we have

$$\mathrm{F}_1 \lesssim \frac{d(x, x')}{\sqrt{t}} \frac{1}{V_{\sqrt{t}}(x)} \int_0^{t/2} \int_{\mathbb{G}} V(z) \frac{1}{V_{\sqrt{s}}(y)} \exp\left\{ -\frac{[d(z,\,y)]^2}{Cs} \right\} \left[\frac{\rho(y)}{\sqrt{s} + \rho(y)} \right]^N d\mu(z)\, ds$$

$$\lesssim \frac{d(x, x')}{\sqrt{t}} \frac{1}{V_{\sqrt{t}}(x)} \left\{ \int_0^{[\rho(y)]^2} \frac{1}{s} \left[\frac{\sqrt{s}}{\rho(y)} \right]^{2-D/q} ds + \int_{[\rho(y)]^2}^{t/2} \frac{1}{s} \left[\frac{\rho(y)}{\sqrt{s}} \right]^{N-\ell_0} ds \right\}$$

$$\lesssim \frac{d(x, x')}{\rho(x)} \frac{1}{V_{\sqrt{t}}(x)}.$$

To estimate F_2, we further write

$$\mathrm{F}_2 \lesssim \int_0^{[d(x,x')]^2} \int_{\mathbb{G}} |\widetilde{T}_s(x,\,z) - \widetilde{T}_s(x',\,z)| V(z) T_{t-s}(z,\,y)\, d\mu(z)\, ds$$

$$+ \int_{[d(x,x')]^2}^{t/2} \int_{W_1} \cdots + \int_{[d(x,x')]^2}^{t/2} \int_{W_2} \cdots \equiv \mathrm{H}_1 + \mathrm{H}_2 + \mathrm{H}_3,$$

where $W_1 \equiv \{z \in \mathbb{G}: \ d(x, x') > d(x, z)/4\}$ and $W_2 \equiv \mathbb{G} \setminus W_1$.

Since $d(x, x') \le \rho(x)$ together with (10) implies that $\rho(x') \sim \rho(x)$, by $d(x, z) \sim d(x', z)$, (46), (47) and Lemma 7.3, we obtain

$$\mathrm{H}_1 \lesssim \frac{1}{V_{\sqrt{t}}(y)} \left[\frac{\rho(y)}{\sqrt{t} + \rho(y)} \right]^N \int_0^{[d(x,x')]^2} \int_{\mathbb{G}} \frac{1}{V_{\sqrt{s}}(x)} \exp\left\{ -\frac{[d(z,\,x)]^2}{Cs} \right\} V(z)\, d\mu(z)\, ds$$

$$\lesssim \frac{1}{V_{\sqrt{t}}(y)} \left[\frac{\rho(y)}{\sqrt{t}+\rho(y)}\right]^N \left[\frac{d(x,x')}{\rho(y)}\right]^{2-D/q} \left[\frac{\rho(y)}{\rho(x)}\right]^{2-D/q}.$$

Recall (2.4) in [31] that for any fixed $y \in \mathbb{G}$ and $0 < r < R < \infty$,

$$\frac{r^2}{V_r(y)} \int_{B(y,r)} V(z)\,d\mu(z) \lesssim \left(\frac{r}{R}\right)^{2-D/q} \frac{R^2}{V_R(y)} \int_{B(y,R)} V(z)\,d\mu(z).$$

Let κ be as in (45). By the inequality above, the assumption that $q \in (D/2, D]$, (47) and Lemma 7.1, we have

$$\mathrm{H}_2 \lesssim \frac{1}{V_{\sqrt{t}}(y)} \left[\frac{\rho(y)}{\sqrt{t}+\rho(y)}\right]^N \int_{[d(x,x')]^2}^{t/2} \int_{W_1} \frac{d(x,x')}{\sqrt{s}} \frac{1}{V_{\sqrt{s}}(x)} V(z)\,d\mu(z)\,ds$$

$$\lesssim \frac{1}{V_{\sqrt{t}}(y)} \left[\frac{\rho(y)}{\sqrt{t}+\rho(y)}\right]^N \left[\frac{d(x,x')}{\rho(y)}\right]^{2-D/q} \left[\frac{\rho(y)}{\rho(x)}\right]^{2-D/q}.$$

For H_3, if $t \le 2[\rho(x)]^2$, by $d(x,x') \le d(x,z)/2$, (47), the assumption that $q \in (D/2, D]$, Lemma 7.1 and Lemma 7.3 with $\sqrt{s} \le \sqrt{2}\rho(x)$, we obtain

$$\mathrm{H}_3 \lesssim \frac{1}{V_{\sqrt{t}}(y)} \left[\frac{\rho(y)}{\sqrt{t}+\rho(y)}\right]^N \int_{[d(x,x')]^2}^{t/2} \int_{W_2} \frac{d(x,x')}{\sqrt{s}} \frac{V(z)}{V_{\sqrt{s}}(x)} \exp\left\{-\frac{[d(z,x)]^2}{Cs}\right\}\,d\mu(z)\,ds$$

$$\lesssim \frac{1}{V_{\sqrt{t}}(y)} \left[\frac{\rho(y)}{\sqrt{t}+\rho(y)}\right]^N \left[\frac{d(x,x')}{\rho(y)}\right]^{\delta'} \left[\frac{\rho(y)}{\rho(x)}\right]^{\delta'}.$$

If $t > 2[\rho(x)]^2$, similarly to the above estimate, using Lemma 7.3 with $\sqrt{s} > \rho(x)$, we have

$$\mathrm{H}_3 \lesssim \frac{1}{V_{\sqrt{t}}(y)} \left[\frac{\rho(y)}{\sqrt{t}+\rho(y)}\right]^N \left\{\int_{[d(x,x')]^2}^{[\rho(x)]^2} + \int_{[\rho(x)]^2}^{t/2}\right\}$$

$$\times \int_{W_2} \frac{d(x,x')}{\sqrt{s}} \frac{1}{V_{\sqrt{s}}(x)} \exp\left\{-\frac{[d(z,x)]^2}{Cs}\right\} V(z)\,d\mu(z)\,ds$$

$$\lesssim \frac{1}{V_{\sqrt{t}}(y)} \left[\frac{\rho(y)}{\sqrt{t}+\rho(y)}\right]^{N-\ell_0+1} \left[\frac{d(x,x')}{\rho(y)}\right]^{2-D/q} \left[\frac{\rho(y)}{\rho(x)}\right]^{\ell_0}.$$

Let k_0 be as in Definition 2.2. Observing that

$$\frac{\rho(y)}{\rho(x)} \lesssim \left[1+\frac{d(x,y)}{\sqrt{t}}\frac{\sqrt{t}}{\rho(y)}\right]^{k_0} \lesssim \left[1+\frac{\sqrt{t}}{\rho(y)}\right]^{k_0} \left[1+\frac{d(x,y)}{\sqrt{t}}\right]^{k_0},$$

we obtain (48) by taking N large enough. The estimate (48) together with Lemma 7.2 implies the desired estimate, which completes the proof of Lemma 7.4. $\qquad\square$

Lemma 7.5. *If $q \in (D/2, D]$ and $V \in \mathcal{B}_q(\mathbb{G}, d, \mu)$, then for all $N \in (0, \infty)$ and $\delta' \in (0, 2 - D/q)$, there exist positive constants $\widetilde{C}$ and C, where C is independent of N, such that for all $t \in (0, \infty)$ and $x', x, y \in \mathbb{G}$ with $d(x, x') < \sqrt{t}$,*

$$|T_t(x, y) - T_t(x', y)| \le \widetilde{C} \left[\frac{d(x, x')}{\sqrt{t}} \right]^{\delta'} \frac{1}{V_{\sqrt{t}}(x)} \exp \left\{ -\frac{[d(x, y)]^2}{Ct} \right\}$$

$$\times \left[\frac{\rho(y)}{\sqrt{t} + \rho(y)} \right]^N \left[\frac{\rho(x)}{\sqrt{t} + \rho(x)} \right]^N. \tag{49}$$

Proof. Let $x', x, y \in \mathbb{G}$ with $d(x, x') < \sqrt{t}$. We first consider the case $d(x, x') \le d(x, y)/4$. Using (3), we obtain

$$\frac{\rho(x')}{\sqrt{t} + \rho(x')} \lesssim \left[\frac{\rho(x)}{\sqrt{t} + \rho(x)} \right]^{1/(1+k_0)} \left[1 + \frac{d(x, y)}{\sqrt{t}} \right]^{k_0/(1+k_0)}. \tag{50}$$

Since $d(x, y) \sim d(x', y)$, by Lemma 7.1, we have

$$|T_t(x, y) - T_t(x', y)| \lesssim \frac{1}{V_{\sqrt{t}}(x)} \exp \left\{ -\frac{[d(x, y)]^2}{Ct} \right\} \left[\frac{\rho(y)}{\sqrt{t} + \rho(y)} \right]^N \left[\frac{\rho(x)}{\sqrt{t} + \rho(x)} \right]^N. \tag{51}$$

If $d(x, x') \ge \rho(x)$, then (49) follows from (51). If $d(x, x') < \rho(x)$, then by Lemma 7.4, we obtain

$$|E_t(x, y) - E_t(x', y)| \lesssim \left[\frac{d(x, x')}{\sqrt{t}} \right]^{\delta'} \left[\frac{\sqrt{t}}{\rho(y)} \right]^{\delta'} \frac{1}{V_{\sqrt{t}}(x)} \exp \left\{ -\frac{[d(x, y)]^2}{Ct} \right\}.$$

This together with (47) and $T_t = \widetilde{T}_t - E_t$ gives that

$$|T_t(x, y) - T_t(x', y)| \lesssim \left[\frac{d(x, x')}{\sqrt{t}} \right]^{\delta'} \left[1 + \frac{\sqrt{t}}{\rho(y)} \right]^{\delta'} \frac{1}{V_{\sqrt{t}}(x)} \exp \left\{ -\frac{[d(x, y)]^2}{Ct} \right\}.$$

Then (49) follows from this and (51).

Now we assume that $d(x, x') \ge d(x, y)/4$. In this case, $d(x, y) < 4\sqrt{t}$. Write

$$|T_t(x, y) - T_t(x', y)| \lesssim \int_{\mathbb{G}} |T_{t/2}(x, z) - T_{t/2}(x', z)| T_{t/2}(z, y) \, d\mu(z)$$

$$\lesssim \int_{W_1} |T_{t/2}(x, z) - T_{t/2}(x', z)| T_{t/2}(z, y) \, d\mu(z) + \int_{W_2} \cdots \equiv \mathrm{I}_1 + \mathrm{I}_2,$$

where W_1 and W_2 are as in Lemma 7.4.

By Lemma 7.1, we have

$$\mathrm{I}_1 \lesssim \left[\frac{V(x, x')}{V_{\sqrt{t}}(x)} + \frac{V(x, x')}{V_{\sqrt{t}}(x')} \right] \frac{1}{V_{\sqrt{t}}(y)} \left[\frac{\rho(y)}{\sqrt{t} + \rho(y)} \right]^N \lesssim \left[\frac{d(x, x')}{\sqrt{t}} \right]^{\delta''} \frac{1}{V_{\sqrt{t}}(y)} \left[\frac{\rho(y)}{\sqrt{t} + \rho(y)} \right]^N,$$

where $\delta'' \in (\delta', 2 - D/q)$.

Using (49) with $d(x, x') \leq d(x, z)/4$ and Lemma 7.1, we obtain

$$\mathrm{I}_2 \lesssim \left[\frac{d(x, x')}{\sqrt{t}}\right]^{\delta''} \frac{1}{V_{\sqrt{t}}(x)} \int_{W_2} T_{t/2}(z, y)\, d\mu(z) \lesssim \left[\frac{d(x, x')}{\sqrt{t}}\right]^{\delta''} \frac{1}{V_{\sqrt{t}}(x)} \left[\frac{\rho(y)}{\sqrt{t} + \rho(y)}\right]^N.$$

By $d(x, y) \leq 4\sqrt{t}$, (50) and Lemma 7.1, we have (49) for $d(x, x') \geq d(x, y)/4$, which completes the proof of Lemma 7.5. $\qquad\square$

For all $x, y \in \mathbb{G}$ and $t \in (0, \infty)$, define

$$Q_t(x, y) \equiv t^2 \frac{d}{ds}\Big|_{s=t^2} T_s(x, y).$$

Following the proof of Proposition 4 in [9], we have the following result. We omit the details.

Lemma 7.6. *Let $q \in (D/2, D]$, $\beta \in (0, 2 - D/q)$ and $N \in \mathbb{N}$. There exist positive constants $\widetilde{C}$ and C, where C is independent of N, such that for all $t \in (0, \infty)$ and $x, x', y \in \mathbb{G}$ with $d(x, x') \leq \frac{t}{2}$,*

(i) $|Q_t(x, y)| \leq \widetilde{C} \frac{1}{V_t(x)} \exp\{-\frac{[d(x,y)]^2}{Ct^2}\} [\frac{\rho(x)}{t+\rho(x)}]^N [\frac{\rho(y)}{t+\rho(y)}]^N$;

(ii) $|Q_t(x, y) - Q_t(x', y)| \leq \widetilde{C} [\frac{d(x,x')}{t}]^\beta \frac{1}{V_t(x)} \exp\{-\frac{[d(x,y)]^2}{Ct^2}\} [\frac{\rho(x)}{t+\rho(x)}]^N [\frac{\rho(y)}{t+\rho(y)}]^N$;

(iii) $|\int_{\mathcal{X}} Q_t(x, y) d\mu(y)| \leq \widetilde{C} [\frac{t}{\rho(x)}]^{2-D/q} [\frac{\rho(x)}{t+\rho(x)}]^N$.

Remark 7.1. *Let $q_1, q_2 \in (D/2, \infty]$ with $q_1 < q_2$. Recall that $\mathcal{B}_{q_2}(\mathbb{G}) \subset \mathcal{B}_{q_1}(\mathbb{G})$. Therefore, Lemmas 7.1 through 7.6 hold for all $q \in (D/2, \infty]$.*

Observe that $\{\widetilde{T}_{t^2}\}_{t>0}$ is a continuous $(1, N)$-AOTI for all positive constants N. Thus $\{T_{t^2}\}_{t>0}$ and $\{\widetilde{T}_{t^2}\}_{t>0}$ satisfy the assumption (22). Moreover, the $L^2(\mathbb{G})$-boundedness of g-function can be obtained by the same argument as in Lemma 3 of [9]. Using these facts and applying Theorems 5.1, 5.2 and 6.1, and Corollaries 5.1, 5.2 and 6.1, we have the following conclusions.

Proposition 7.5. *Let $q \in (D/2, \infty]$, $V \in \mathcal{B}_q(\mathbb{G}, d, \mu)$ and ρ be as in (4). There exists a positive constant C such that for all $f \in \mathrm{BMO}_\rho(\mathbb{G})$, $T^+(f)$, $\widetilde{T}^+(f)$, $\widetilde{T}^+_\rho(f)$, $P^+(f)$, $\widetilde{P}^+(f)$, $\widetilde{P}^+_\rho(f)$, $g(f)$, $[g(f)]^2 \in \mathrm{BLO}_\rho(\mathbb{G})$ and*

$$\|T^+(f)\|_{\mathrm{BLO}_\rho(\mathbb{G})} + \|\widetilde{T}^+(f)\|_{\mathrm{BLO}_\rho(\mathbb{G})} + \|T^+_\rho(f)\|_{\mathrm{BLO}_\rho(\mathbb{G})} + \|P^+(f)\|_{\mathrm{BLO}_\rho(\mathbb{G})}$$
$$+ \|\widetilde{P}^+(f)\|_{\mathrm{BLO}_\rho(\mathbb{G})} + \|\widetilde{P}^+_\rho(f)\|_{\mathrm{BLO}_\rho(\mathbb{G})} + \|g(f)\|_{\mathrm{BLO}_\rho(\mathbb{G})} + \|[g(f)]^2\|_{\mathrm{BLO}_\rho(\mathbb{G})}^{1/2}$$
$$\leq C\|f\|_{\mathrm{BMO}_\rho(\mathbb{G})}.$$

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E-mail address: dcyang@bnu.edu.cn

E-mail address: dyyang@mail.bnu.edu.cn

E-mail address: yuanzhou@mail.bnu.edu.cn